An arithmetic count of lines meeting four lines

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Some enumerative problems in algebraic geometry

Fix a field k.

- How many zeroes does a degree d polynomial have?
- How many lines lie on a smooth cubic surface?
- How many lines meet four lines in $\mathbb{P}^3$?

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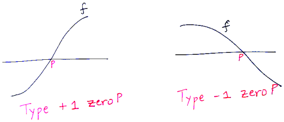
Question 1: What if $k \neq \mathbb{C}$?

Question 2: Extra arithmetic-geometric data when $k \neq \mathbb{C}$?

Assume d even.
Let $f \in \mathbb{R}[x]$ of degree d, all zeroes multiplicity 1 and real.



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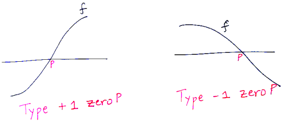
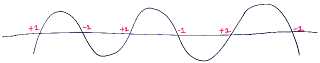


Observe: # Type 1 real zeroes = # Type -1 real zeroes



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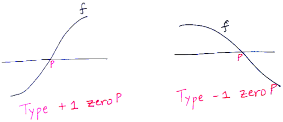

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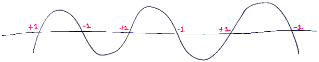

$\mathbb{A}^1$ – reformulation: $$\sum_{P:f(P)=0} \mathrm{Type}(P) = \frac{d}{2}(\langle 1\rangle + \langle -1\rangle) \in \mathrm{GW}(\mathbb{R})$$

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Remark: $W(k) = \mathrm{GW}(k)/(h)$

$$\mathrm{GW}(\mathbb{C}) \xrightarrow[\text{rank/dimension}]{\sim} \mathbb{Z}$$

$$\mathrm{GW}(\mathbb{R}) \xhookrightarrow[\text{rank,signature}]{} \mathbb{Z} \times \mathbb{Z}$$

$$\mathrm{GW}(\mathbb{F}_q) \xrightarrow[\text{rank, discriminant}]{\sim} \mathbb{Z} \times \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Grothendieck-Witt groups and field extensions

Extension: If $k \subset L$ is an extension of fields, then we have a map

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Trace/Transfer: For $k \subset L$ a finite separable extension, we have

$$\begin{aligned} \mathrm{Tr}_{L/k}\colon\ \mathrm{GW}(L) &\to \mathrm{GW}(k) \\ (V \times V \to L) &\mapsto (V \times V \to L \xrightarrow{\mathrm{Tr}_{L/k}} k) \end{aligned}$$

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Question:
Is there an enriched $c^{\mathrm{top}}(\mathcal{V}) \in \mathrm{GW}(k)$ when $\dim(X) = \mathrm{rank}(\mathcal{V})$?
Yes! If $\mathcal{V}$ is relatively oriented.

Geometric type of a line meeting four lines

Let L be a line meeting four lines L_1, L_2, L_3, L_4.
Assume all lines are generic and that $\text{char}(k) \neq 2$.

Question: Can we define $\text{Type}(L) \in \text{GW}(k)$ using geometric data?

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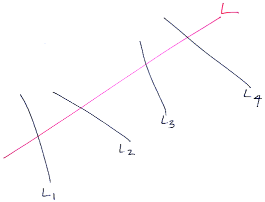
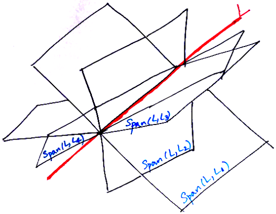
Assume all lines are generic and that $\mathrm{char}(k) \neq 2$.

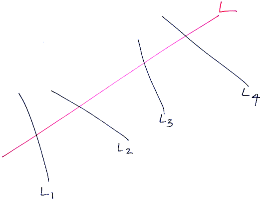
Question: Can we define $\mathrm{Type}(L) \in \mathrm{GW}(k)$ using geometric data?
Answer: Yes! $\mathrm{Type}(L) = \mathrm{Tr}_{k(L)/k}\langle \lambda_L - \mu_L \rangle$.

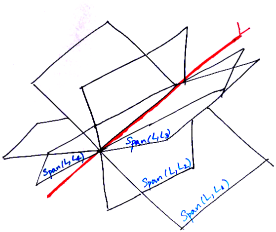




$\lambda_L = $ Cross-ratio of $L \cap L_i$ $\quad$ $\mu_L = $ Cross-ratio of $\mathrm{Span}(L, L_i)$

An arithmetic count of lines meeting four lines





$$\mathrm{Type}(L) = \mathrm{Tr}_{k(L)/k}\langle \lambda_L - \mu_L \rangle.$$

Theorem (S-Wickelgren)

Assuming that all four lines L_i are defined over k, we have

$$\sum_{L : L \cap L_i \neq \emptyset \ \forall i} \mathrm{Type}(L) = \langle 1 \rangle + \langle -1 \rangle \in \mathrm{GW}(k)$$

$X = \mathrm{Gr}(2,4), \mathcal{V} = \sum_{i=1}^{4} \Lambda^2 \mathcal{S}^\vee$
σ a rational section of $\mathcal{V}$ coming from L_1, L_2, L_3, L_4.

Kass-Wickelgren: $e(\mathcal{V}) = \sum_{P:\sigma(P)=0} \deg_P(\sigma) \in \mathrm{GW}(k)$.

Proof Sketch:

- Show for special choice of explicit section σ of $\mathcal{V}$, we have

$$\deg_L(\sigma) = \mathrm{Tr}_{k(L)/k}\langle \lambda_L - \mu_L \rangle.$$

- If L and L' are the two lines that meet all four lines, then

$$(\lambda_L, \mu_L) = (\mu_{L'}, \lambda_{L'}).$$

- Use $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle$.

Claim: If $k = \mathbb{Q}$, there does not exist a pair L, L' of 2 conjugate lines defined over $\mathbb{Q}(\sqrt{3})$ meeting four lines defined over $\mathbb{Q}$ with $\langle \lambda_L - \mu_L \rangle = \langle 5 \rangle$.

Main Theorem $\Rightarrow \mathrm{Tr}_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}} \langle 5 \rangle = \langle 1 \rangle + \langle -1 \rangle$.

Left hand side: In the $\mathbb{Q}$-basis $(1, \sqrt{3})$ for $\mathbb{Q}(\sqrt{3})$, the matrix for the bilinear form $\mathrm{Tr}_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}} \langle 5 \rangle$ is

$$\begin{bmatrix} 2 \cdot 5 & 0 \\ 0 & 2 \cdot 5 \cdot 3 \end{bmatrix}$$

$\Rightarrow \mathrm{disc}(\mathrm{Tr}_{\mathbb{Q}(\sqrt{3})/\mathbb{Q}} \langle 5 \rangle) = 300$.

Right hand side: $\mathrm{disc}(\langle 1 \rangle + \langle -1 \rangle) = -1$.

Contradiction: $300 \neq -1$ in $\mathbb{Q}^* / (\mathbb{Q}^*)^2$.