The section conjecture at the boundary of $\overline{\mathcal{M}_g}$

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Outline

k: finitely generated field of characteristic 0

$\overline{k}$: a separable closure of k

X: smooth projective geometrically integral k-curve of genus at least 2

$\overline{x}$: geometric point of X, i.e., $\overline{x} \in X(\overline{k})$

Then there is an exact sequence of étale fundamental groups.

$$\pi_1\text{-seq}(X): \qquad 1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \text{Gal}(\overline{k}/k) \to 1.$$

Conjecture (Grothendieck)

The natural map $X(k) \xrightarrow{\text{sec}} \text{Splittings of } \pi_1\text{-seq}(X)/\text{conjugacy}$ *is a bijection.*

Obstructions to splittings, and hence rational points

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Main Question: (*"Trivial" case of the section conjecture*)
Are there examples of curves X and fields k where π_1-seq(X) does not split?

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Known examples of the trivial case of the section conjecture:

- *(Arithmetic, local)* Stix, 2008. $k = \mathbb{Q}_p$. Local p-adic obstructions to sections.
- *(Arithmetic, global)* Harari and Szamuely, 2009. k number field. Counterexamples to the Hasse principle for sections.
- *(Geometric)* Hain, 2011.
  $k = \mathbb{C}(\mathcal{M}_g)$, $X = \mathcal{C}_g$ universal genus g curve, $g \geqslant 5$.

Related exact sequences split by rational points

Lower central series filtration on $\pi := \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x})$:

$$\pi \supset L^2\pi := \overline{[\pi, \pi]} \supset L^3\pi := \overline{[\pi, L^2\pi]} \supset L^4\pi \supset \cdots .$$

π_1-seq(X):

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

π_1-ab-seq(X):

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x})^{\text{ab}} \to \pi_1^{\text{ét}}(X, \bar{x})/L^2\pi \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

π_1-2nil-seq(X):

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x})/L^3\pi \to \pi_1^{\text{ét}}(X, \bar{x})/L^3\pi \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

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π_1-seq(X): weakest nonsplitting results

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

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π_1-2nil-seq(X): intermediate

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, \bar{x})/L^3\pi \to \pi_1^{\text{ét}}(X, \bar{x})/L^3\pi \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

New geometric and arithmetic examples that trivially satisfy the section conjecture

Theorem (Li, Litt, Salter, S.)

1. *Geometric (abelian obstruction for the universal curve)*
   $k = \mathbb{C}(\mathcal{M}_g)$, $X = \mathcal{C}_g$ *universal genus* g *curve,* $g \geqslant 3$.
   π_1-ab-seq(X) *does not split.*

2. *Geometric (2-nilpotent obstruction for the pullback of the universal curve to the moduli space of degree 1 divisors)*
   $k = \mathbb{C}(\mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g})$, $X = f^*(\mathcal{C}_g)$, *along* $f\colon \mathrm{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g} \to \mathcal{M}_g$,
   g *even.* π_1-2nil-seq(X) *does not split.*

3. *Arithmetic*
   - *There are infinitely many curves* X *over p-adic fields such that the sequence* π_1-ab-seq(X) *does not split.*
   - *There are infinitely many curves* X *over p-adic fields such that the sequence* π_1-ab-seq(X) *splits but the sequence* π_1-2nilp-seq(X) *does not split.*

Outline

$\mathcal{M}_g/k$:	moduli space of genus g curves over a field k, $g \geqslant 3$
$\mathcal{C}_g/\mathcal{M}_g$:	universal genus g curve over $\mathcal{M}_g/k$
$\overline{\mathcal{M}_g}$:	Deligne-Mumford compactification of $\mathcal{M}_g$
Γ:	stable graph indexing a boundary stratum of $\overline{\mathcal{M}_g}$
Z_Γ:	boundary stratum corresponding to Γ
$\widehat{K_\Gamma}$:	completion of $k(\mathcal{M}_g)$ along the (blowup E_Γ of) Z_Γ
$\mathcal{C}_{g,\widehat{K_\Gamma}}$:	$\mathcal{C}_g \otimes_{k(\mathcal{M}_g)} \widehat{K_\Gamma}$, "universal curve with reduction type Γ"

Conjecture (Li, Litt, Salter, S.)

For every field k and stable graph Γ, the curve $\mathcal{C}_{g,\widehat{K_\Gamma}}$ ("the universal curve with reduction type Γ") trivially satisfies the section conjecture.

Remark: If the boundary stratum $Z_{\Gamma'}$ is in the closure of Z_Γ and if the tropical section is true for Γ', then it is also true for Γ.

The tropical section conjecture is true for some stable graphs

Theorem (Li, Litt, Salter, S.)

Let k be a characteristic 0 field. The tropical section conjecture is true for the following graphs.

- *For $g \geqslant 3$, let H_g be the stable graph consisting of a $g-1$-cycle, all of whose vertices have genus 1.*

  The sequence π_1-ab-seq$(\mathcal{C}_{g,\widehat{K_{H_g}}})$ does not split for H_g.

- *For $g \geqslant 2$ even, let T_g be the stable graph consisting of two vertices of genus $g/2$ connected by an edge.*

  The sequence π_1-2nilp-seq$(\mathcal{C}_{g,\widehat{K_{T_g}}})$ does not split for T_g.

Proof Sketch

1. Construct obstruction classes

   There is an $o_1 \in H^2(k(\mathcal{M}_g), \pi_1^{\text{ét}}(\overline{\mathcal{C}_g})^{\text{ab}})$ that obstructs splitting of the abelian sequence for the universal curve over $k(\mathcal{M}_g)$.

   There is an $o_2 \in \overline{H^2(k(\mathbf{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g}), L^2\pi/L^3\pi)}$ that obstructs splitting of the 2-nilpotent sequence for the universal curve over $k(\mathbf{Pic}^1_{\mathcal{C}_g/\mathcal{M}_g})$.

2. Show nonvanishing of obstruction classes when $k = \mathbb{C}$

   Pullback to a tubular neighbourhood of a boundary component over $\mathbb{C}$.
   Explicitly compute orders of o_1 and o_2 after pullback, and show they are nonzero. (Topological computation)

Comparing $o_{1,\mathbb{C}}$ with Morita's obstruction class

Spread out the exact sequence of fundamental groups from the generic point to the whole moduli space $\mathcal{M}_g$.

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Σ_g : genus g compact orientable surface

Mod_g : Mapping class group of $\Sigma_g (= \pi_1(\mathcal{M}_g))$

Birman exact sequence of fundamental groups for the fibration

$$\Sigma_g \to \mathcal{C}_{g,\mathbb{C}} \to \mathcal{M}_{g,\mathbb{C}}$$

$$(*) \quad 1 \to \pi_1(\Sigma_g) \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g \to 1.$$

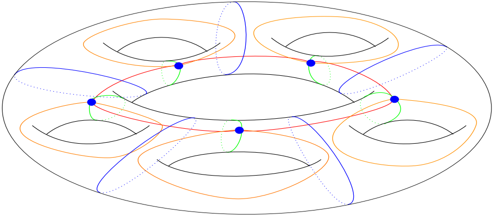
Morita studied the obstruction class for splitting $(*)$ after pushout by $\pi_1(\Sigma_g) \to \pi_1(\Sigma_g)^{\mathrm{ab}}$.

A topological computation of obstruction classes over $\mathbb{C}$

$$(*) \quad 1 \to \pi_1(\Sigma_g) \to \mathrm{Mod}_{g,1} \to \mathrm{Mod}_g \to 1.$$

Our strategy:

We pushout (*) along $\pi_1(\Sigma_g) \to \pi_1(\Sigma_g)^{\mathrm{ab}}$ and pullback along $\mathbb{Z}^2 \to \mathrm{Mod}_g$ to get a *computable* class $o_1|_{\dots} \in H^2(\mathbb{Z}^2, \pi_1(\Sigma_g)^{\mathrm{ab}})$.



The Riemann Surface Σ_6 with the two mapping classes giving $\mathbb{Z}^2 \to \mathrm{Mod}_6$
The blue curves give rise to a multi-Dehn twist giving a mapping class
The other mapping class is a rotation of order 5 permuting the blue curves

From geometric to arithmetic examples for the trivial case of the section conjecture

Want: Infinitely many closed points $[C] \in \mathcal{M}_g$ such that $o_1|_C \neq 0$.

Proof sketch continued

3. Lower cohomological degree from H^2 to H^1 using Gysin maps

   The boundary strata Γ give rise to Gysin maps, for e.g.,

$$H^2(\widehat{K_\Gamma}, H^1(\Sigma_g, \hat{\mathbb{Z}})) \xrightarrow{g_{\Gamma,1}} H^1(k(E_\Gamma), H^1(\Sigma_g, \hat{\mathbb{Z}})_{I_\Gamma})$$

   Show $g_{\Gamma,1}(o_1) \neq 0$ and $g_{\Gamma,2}(o_2) \neq 0$.

4. Prove a variant of the Chebotarev density theorem for H^1

   Show infinitely many specializations of $g_{\Gamma,1}(o_1)$ and $g_{\Gamma,2}(o_2)$ to closed points also do not vanish.

   $\rightsquigarrow$ Infinitely many curves over p-adic fields with no sections.