p-ADIC ADELIC METRICS AND QUADRATIC CHABAUTY I

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Abstract. We give a new construction of p-adic heights on varieties over number fields using p-adic Arakelov theory. In analogy with Zhang's construction of real-valued heights in terms of adelic metrics, these heights are given in terms of p-adic adelic metrics on line bundles. In particular, we describe a construction of canonical p-adic heights an abelian varieties and we show that, for Jacobians, this recovers the height constructed by Coleman and Gross. Our main application is a new and simplified approach to the Quadratic Chabauty method for the computation of rational points on certain curves over the rationals, by pulling back the canonical height on the Jacobian with respect to a carefully chosen line bundle.

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1. Introduction

The explicit solution of polynomial equations in rationals or integers is one of the oldest problems in mathematics. Even the simplest case, that of finding rational (or integral) points on a smooth projective algebraic curve X over the rationals, does not have a satisfactory solution except in special cases. If the genus of X is at least 2, then one knows, thanks to the celebrated theorem of Faltings [Fal83], that the set $X(\mathbb{Q})$ of rational points is finite. But the theorem does not give an effective way of bounding its size, let alone a computationally feasible way of finding it.

Chabauty's theorem [Cha41], made effective by Coleman [Col85a], raised the hope of finding such a method using p-adic methods. If the Mordell-Weil rank $r = \operatorname{rk}(J/\mathbb{Q})$ of the Jacobian $J = J_X$ of X is smaller than the genus g of X, it shows that $X(\mathbb{Q}) \subset X(\mathbb{Q}_p)$ is contained in the zero set of a locally analytic function that can often be made explicit. Chabauty proved that this zero set is finite. When p is a prime of good reduction, Coleman identified the relevant function as an example of a p-adic line integral in the sense of [Col85b] and gave an effective bound on the number of points. The method of Chabauty-Coleman can often be used to compute $X(\mathbb{Q})$ in practice, by explicitly computing Coleman integrals. See [MP12] for an exposition. One should note that all of the above and, at least in principle, all that follows, extends to the study of K-rational points on a curve X/K, where K is a number field; see for instance [Sik13] and [BBBM21].

The revolutionary paper of Kim [Kim05] outlined a program to push the method of Chabauty and Coleman to the case of curves which do not satisfy the Chabauty bound $r < g$. There it was demonstrated for the first time that one may use the arithmetic theory of the fundamental group to identify more general Coleman integrals, namely, iterated integrals, which would vanish on $X(\mathbb{Q})$. Shortly afterwards, Kim conjectured [Kim09] that one could in fact recover $X(\mathbb{Q})$ completely using these methods. Verifying these conjectures is the main open problem of the subject. Nevertheless, a more practical problem is to recover $X(\mathbb{Q})$ using p-adic methods, and being slightly more restrictive and precise, trying to find $X(\mathbb{Q})$ inside the zero set of some iterated Coleman integral (for different approaches toward making the theorem of Faltings effective that do not use Coleman integrals and that are indeed effective in some cases, see [LV20] and [Alp20]): One method of this sort is Quadratic Chabauty [BBM16, BBM17, BBBM21, BD18, BD21, DF21, BDM$^+$19, BBB$^+$21, BDM$^+$21, AAB$^+$21, EL21, ČLYX21].

Stated broadly (but still not covering the method of geometric Quadratic Chabauty [EL21, ČLYX21] on which we have nothing to say in this work), Quadratic Chabauty can be phrased as an equality of functions on $J(\mathbb{Q})$ of the form $Q = H$. Here, Q is in fact a local function

$$Q\colon J(\mathbb{Q}_p) \to \mathbb{Q}_p \,.$$

In contrast, H, which usually comes from some sort of a p-adic height, is global in nature. Fixing an Abel-Jacobi embedding $\iota\colon X \to J$ over $\mathbb{Q}$, one has a non-canonical decomposition

$$H \circ \iota = \sum_{v<\infty} \lambda_v \ ,$$

where v runs through all finite places of $\mathbb{Q}$ and λ_v is a function $X(\mathbb{Q}_v) \to \mathbb{Q}_p$. To obtain the equality $Q = H$ one first proves that H is a quadratic function. One may then use this to write H in terms of a basis of such functions which can be extended to $J(\mathbb{Q}_p)$. For instance, one can write as a quadratic polynomial in the linear functionals on $J(\mathbb{Q})$ coming from Coleman integrals of holomorphic forms. By pulling back along ι, one obtains an equation, satisfied for $x \in X(\mathbb{Q})$,

$$Q \circ \iota(x) - \lambda_p(x) = \sum_{q<\infty,\, q\neq p} \lambda_q(x) \,. \tag{1}$$

Here, the left hand side is a locally analytic function $X(\mathbb{Q}_p) \to \mathbb{Q}_p$ with finite fibers. The remaining ingredient of Quadratic Chabauty is that H needs to be constructed in such a way that the right hand side of the above equation takes values in a finite computable set T. Typically, λ_q is going to be identically 0 for all primes of good reduction and for primes of bad reduction it can take a finite set of values.

In the first instance of Quadratic Chabauty [BBM16, BBM17] the curve X was hyperelliptic of odd degree with $r = g$, the embedding ι was done via the point at infinity and the function H was simply the p-adic height. However, while (1) was satisfied for $x \in X(\mathbb{Q})$, the method only worked for integral points (with respect to an odd degree equation with coefficients in $\mathbb{Z}$). The reason was that $\lambda_q(x)$ can be expressed as an intersection multiplicity on a regular model of X, and it was only possible to control this multiplicity for q-integral points $x \in X(\mathbb{Q}_q)$.

The next substantial step was taken by Balakrishnan and Dogra in [BD18]. This important paper simultaneously related the theory of [BBM16] with the arithmetic of the fundamental group of X, by constructing H using an extension of the Nekovář height pairing [Nek93] on an appropriate nonabelian unipotent quotient of the fundamental group, and, more importantly, showed how additional geometric data such as a non-trivial correspondence on X, could allow the method to be extended to recover the rational points and not just the integral points. This extension of Quadratic Chabauty has been used since to find the rational points in a number of cases that resisted resolution using previous methods, most importantly in the case of the cursed curve [BDM+19] related to Serre's uniformity problem about the image of Galois representations associated to elliptic curves. Other modular examples have been computed using this method in [BBB+21, AAB+21, BDM+21]. It is important to note that the most non-trivial part of the extension to Quadratic Chabauty in [BD18] and subsequent work such as [BD21] is the finiteness property. Indeed, given a correspondence on X one can simply compose the p-adic height with that correspondence and obtain a new quadratic form, but in general, there is no hope that this will satisfy the finiteness property when pulled back to X.

The paper [BD18] makes full use of both the theory of the fundamental group a la Kim and Nekovář's cohomological height pairing. In the end though, the resulting function H is fairly simple, and when pulled back to X can be expressed as the height pairing applied to two divisors constructed out of a point $x \in X(\mathbb{Q})$ and the correspondence in a very explicit way. A natural problem that arises is thus to explain the results of [BD18] in a more direct way without the detour into the theory of the fundamental group. This problem was the starting point of the present work. We provide a solution based on p-adic Arakelov theory. One important consequence is that the function λ_p, which one needs to compute and even locally expand as a power series in order the make the method work, is now (essentially) explicitly given as an iterated Coleman integral. Another side effect is that the theory extends without any further work to the case of primes p of bad reduction, using the theory of Vologodsky integration [Vol03] instead of Coleman integration.

To give our version of Quadratic Chabauty, we replace the use of the Nekovář construction of the height pairing by a new p-adic analogue of the theory of adelic metrics and associated heights due to Zhang [Zha95] which we believe could be of independent interest. Let us recall that an adelic metric on a line bundle L over a variety X over a number field K assigns to each (finite or infinite) place v of K a real-valued norm $\|\cdot\|_v$ on the completion L_v, compatible with a set of absolute values on K satisfying the product formula and subject to some natural conditions. These guarantee, among other things, that the associated height

$$h_L(x) = \sum_v \log \|u\|_v \in \mathbb{R}\,,$$

where u is any nonzero vector in the fiber of L above x, is a finite sum, hence well-defined. This height decomposes non-canonically into local contributions by picking a nonzero section (which could in principle be only set theoretical) s of L, giving

$$h_L(x) = \sum_v \log \|s(x)\|_v \in \mathbb{R}\,. \tag{2}$$

Following Tate's method for obtaining canonical heights on abelian varieties and similar methods for arithmetic dynamical systems, Zhang showed how to obtain canonical adelic metrics for arithmetic dynamical systems. Suppose $f\colon X \to X$ is an endomorphism of a variety X over a number field K, L is a line bundle on X and $\beta\colon L^{\otimes d} \to f^*L$ is an isomorphism, with $d > 1$ (let us refer to the situation above as a dynamical situation). Then, for any place v of K, starting with any norm $\|\cdot\|_v$ on L_v, repeatedly replacing $\|\cdot\|_v$ with $(\beta^{-1}f^*\|\cdot\|_v)^{\frac{1}{d}}$ and taking a limit, we obtain a canonical norm on L_v for which β is an isometry, and these together give a canonical adelic metric. As an example, if $X = (E, P_\infty)$ is an elliptic curve given by a Weierstrass equation with point at infinity P_∞, $f = 2$ is the multiplication by 2 map, $L = \mathcal{O}(2P_\infty)$ and $d = 4$, one obtains an adelic metric on E whose associated height is easily recognized as the standard canonical height on E. In many important cases, for instance when X is an abelian variety and f is multiplication by 2, the canonical norm for a non-archimedean place v is given in the form $\|u\|_v = c^{v_L(u)}$ where $1 > c \in \mathbb{R}$ and v_L is a $\mathbb{Q}$-*valuation* on L_v – a $\mathbb{Q}$-valued function v on L_v^*, the total space of L_v without the zero section. See Definition 2.1; valuations are called log-metrics by Betts in [Bet17].

To construct a p-adic analogue of the theory above, we begin by replacing the system of absolute values satisfying the product formula (more precisely, the log of this system) by an idelé class character

$$\chi = \sum_v \chi_v\colon \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p\ .$$

The logs of the norms on a line bundle are replaced by certain functions $\log_v\colon L_v^* \to \mathbb{Q}_p$. In analogy with logs of norms, these satisfy the relation $\log_v(aw) = \chi_v(a) + \log_v(w)$ for any $a \in K_v$ and nontrivial $w \in L_x$. There is an obvious notion of isometry in the theory; furthermore, the functions $\log_v$ and the associated heights behave functorially.

As is often the case, it is easy to see that the limiting method used to construct canonical norms in the classical theory will not work in the p-adic case and a different method has to be used. For places v which are not above p one can use valuations, such as the ones provided by the classical theory as before, to obtain $\log_v$. Indeed, since for such a place the character χ_v factors via ord_v, it is easy to see that setting $\log_v = \chi_v(\pi_v)\cdot v_L$ gives a function with the desired properties.

Even this method fails for places above the prime p. One of the key observations of the present work is that in this case one may use p-adic Arakelov theory [Bes05] to obtain canonical log functions. More precisely, we use the theory of p-adic log functions and their curvature forms. For simplicity, let us restrict now to the case of the p-adic completion $\mathbb{Q}_p$ of the rationals. The theory isolates among all log functions a class of nice log functions which are integrals (in the sense of Vologodsky) of a particularly simple form. To those log functions on a line bundle L over a variety X over $\mathbb{Q}_p$ the theory associates a curvature form $\alpha \in H^1_{\mathrm{dR}}(X) \otimes \Omega^1(X)$ with the property that $\cup\alpha = \mathrm{ch}_1(L) \in H^2_{\mathrm{dR}}(X)$. Furthermore, the condition about the cup product of α is necessary and sufficient for the construction of a log function on L with curvature α. Unlike the classical archimedean theory, this log function is now no longer uniquely determined by α up to a constant, but can be changed by adding the integral of a form $\omega \in \Omega^1(X)$. This suggests a way of finding a canonical log function for the dynamical situations described above: First find a curvature form α cupping to $\mathrm{ch}_1(L)$ and satisfying $f^*\alpha = d \cdot \alpha$ (Note that this equality will hold after applying the cup product, so when the kernel of the cup product is not so big this is quite reasonable – for instance, for abelian varieties). This makes the isomorphism β an isometry up to the integral of a holomorphic form. Now adjust the log function by the integral of a holomorphic form to make β an isometry.

We consider this procedure in the case that $X = A$ is an abelian variety and f is the multiplication by 2 map. We prove that it gives log functions at places above p that produce heights which are quadratic for symmetric line bundles and linear for antisymmetric line bundles. However, unlike the classical real-valued theory, the resulting log functions are not unique. For a symmetric line bundle L there is a unique good log function for each curvature form cupping to $\mathrm{ch}_1(L)$ whereas for an antisymmetric line bundle, every log function with trivial curvature is good. To get canonical log functions, and consequently canonical heights, we pick an appropriate log function on the Poincaré line bundle on $A \times \hat{A}$. This depends on a choice, well known in the theory of p-adic heights, of a complementary subspace to $\Omega^1(A_v)$ inside $H^1_{\mathrm{dR}}(A_v)$ for every place v above p. From this we obtain canonical adelic metrics and canonical heights for antisymmetric line bundles by restricting to the relevant fiber above $\hat{A}$ and to symmetric line bundles by pulling back via an appropriate map $A \to A \times \hat{A}$.

Having the theory of p-adic heights in place, we have a simple description of Quadratic Chabauty. Let $\iota\colon X \to J$ be an embedding of the curve over $\mathbb{Q}$ into its Jacobian as before. Suppose that there exists a line bundle L on J such that ι^*L is the trivial line bundle $\mathcal{O}_X$ and such that L itself is not algebraically equivalent to 0. The function H is going to be $H = h_L$, the canonical height associated to the line bundle L. Thus, H is a quadratic function by our theory of heights. On the other hand, the local components of the canonical height pull back to functions $\log_v$ on $\mathcal{O}_X$, so we have a decomposition on $X(\mathbb{Q})$:

$$H \circ \iota = \sum_v \lambda_v\,,$$

where $\lambda_v(x) = \log_v \circ 1(x)$ and 1 is the canonical section of $\mathcal{O}_X$. Let $T = \{\sum_{q\neq p} l_q \cdot \chi_q(q)\}$, where l_q runs through the values that the function λ_q takes on $X(\mathbb{Q}_q)$. Our main result is the following:

Theorem 1.1. *The function*

$$F := h_L \circ \iota - \log_v \circ 1\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$$

is locally analytic and takes values on $X(\mathbb{Q})$ in T. Moreover, T is finite and for every $t \in T$, there are only finitely many $x \in X(\mathbb{Q}_p)$ with $F(x) = t$.

See Theorem 7.3 for a slightly more precise statement. If we can make all quantities in Theorem 1.1 explicit, then we obtain a finite subset of $X(\mathbb{Q}_p)$ containing $X(\mathbb{Q})$.

We note that the starting point for the geometric Chabauty method, a nontrivial class in

$$\ker\colon \mathrm{NS}(J) \to \mathrm{NS}(X)$$

is equivalent to having a line bundle such as the one that we need in our method. Hence our requirements for Quadratic Chabauty are precisely the same as in [BD18] and [EL21], but our approach is not restricted to good reduction at p. Moreover, our auxiliary choices are the same as in [BD18].

One advantage of the approach to Quadratic Chabauty presented in this paper is that the function λ_p is just $\log_p(1)$, the log of the section 1 of the trivial bundle with respect to a log function with easily computed curvature form. As explained before, this determines it up to the integral of an unknown holomorphic differential. An important observation is that knowledge of this differential is not needed in order to make the method work. We explain how this function can be explicitly written as a Coleman or Vologodsky iterated integral. In future work, we plan to implement this approach in practice, using algorithms for single [BBK10, Bal15, BT20, KK20, Kay20] and double integrals [Bal13].

We show that, for $q \neq p$, the function λ_q factors through the reduction graph of $X \otimes \mathbb{Q}_q$, by relating it to a Néron function on J with respect to L_q. In particular, λ_q vanishes when X has potentially good reduction at q. It seems difficult to compute the possible values of λ_q in general. This is analogous to the situation in [BD18, BDM+19]; an approach to this problem was developed by Betts and Dogra [BD19], see also [BDM+21, Theorem 3.2]. However, these results do not give an algorithm to compute the local contributions away from p in the setting of [BD18]. In future work, we will explain how Vologodsky integration may be used to construct canonical valuations at $q \neq p$ and how to use this for explicit computations of the possible values of λ_q.

It is a natural question how our p-adic height relates to other constructions in the literature. We show that on any abelian variety, our canonical p-adic height fits into the general framework of p-adic heights pairing due to Mazur-Tate [MT83]. For Jacobians, we show that we recover the pairing constructed by Coleman and Gross [CG89] and hence, via the comparison results of [Bes04, Bes17], the pairing of Nekovář [Nek93] when the curve has semistable reduction at all places above p.

Finally, we discuss the relation between our construction and the one from [BD18]. Via the above-mentioned comparison between our new p-adic height and the one of Nekovář, we show in Proposition 8.5 that the global function H is in fact the same in both constructions up to a constant factor. In future work, we will give a precise comparison formula for the local contributions. It would be interesting to compare our approach with the one of Edixhoven and Lido [EL21].

The outline of the paper is as follows. In Section 2 we reformulate the theory of (logarithms of) metrics over non-archimedean local fields, especially of canonical metrics on abelian varieties, in terms of the notion of valuations. Section 3 summarizes the necessary p-adic Arakelov theory developed in [Bes05], with a focus on the case of curves. We then use this theory to construct canonical log functions on abelian varieties over $\mathbb{Q}_p$ in Section 4, the central section of the paper. The global theory of adelic p-adic metrized line bundles and their associated heights is presented in Section 5. We then give comparison results between our heights and those of Mazur-Tate and (in the case of Jacobians) Coleman-Gross in Section 6. Section 7 contains our construction of Quadratic Chabauty; in particular, we prove Theorem 1.1. Finally, we relate the global functions (coming from p-adic heights) used in our construction to those due to Balakrishnan and Dogra factor in Section 8.

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1.1. **Notation.** We call a variety X over a field K *nice* if it is smooth, projective and geometrically integral. We write $x \in X$ as a shorthand for $x \in X(\overline{K})$, where $\overline{K}$ is an algebraic closure of K. For a line bundle L/X, we let $L^\times$ denote the complement of the zero section in the total space of L. A *rigidification* of L is a choice of an element $r \in L_P(K)$ for some $P \in X(K)$. An *isomorphism of rigidified line bundles* on X is an isomorphism of the underlying bundles that maps the rigidification on one side to the rigidification on the other under the induced map on total spaces. *Tensor products and pullbacks of rigidified line bundles* can analogously be defined. (See [BG06, 9.5.6].)

We fix an embedding of $\mathbb{Q}_p$ into a fixed algebraic closure $\overline{\mathbb{Q}}_p$. Let ord_p be the extension of the discrete valuation of $\mathbb{Q}_p$ to $\overline{\mathbb{Q}}_p$.

2. Valuations and canonical local heights away from p

In this section, $X/\overline{\mathbb{Q}}_p$ denotes a smooth proper variety and L a line bundle on X.

Definition 2.1. A *valuation on* L is a function $v_L \colon L^\times(\overline{\mathbb{Q}}_p) \to \overline{\mathbb{Q}}_p$ such that for every fiber L_x we have

$$v_L(\lambda u) = \mathrm{ord}_p(\lambda) + v_L(u)$$

for every nonzero $u \in L_x$, and every $\lambda \in \overline{\mathbb{Q}}_p$. A $\mathbb{Q}$-*valuation on* L is a valuation on L with values in $\mathbb{Q}$.

Remark 2.2. When L and X are defined over a local field $K \subset \overline{\mathbb{Q}}_p$, the function v_L is invariant under the action of the absolute Galois group of K, as we now explain. Pick a non-vanishing section u in $L_x(K)$. Any point in $L_x(F)$ for some extension field F of K is of the form λu for some $\lambda \in F$, and $\sigma(\lambda u) = \sigma(\lambda)u$. Now

$$v_L(\sigma(\lambda u)) = \mathrm{ord}_p(\sigma(\lambda)) + v_L(u) = \mathrm{ord}_p(\lambda) + v_L(u) = v_L(\lambda u)\,.$$

Example 2.3. Let K be a local field with ring of integers R. Let $\mathcal{L}$ be a line bundle over a proper, flat reduced scheme $\mathfrak{X}$ over R. Let X be the generic fiber of $\mathfrak{X}$, and $L := \mathcal{L}_K$ the corresponding line bundle over X. We now define a valuation $v_{\mathcal{L}}$ on L called the *model valuation* associated to $\mathfrak{X}, \mathcal{L}$.

Let F be a finite extension of K, with ring of integers R_F and $P \in X(F)$. Let s be an invertible meromorphic section of L such that $P \notin \mathrm{div}(s)$. By flatness, there is a unique extension $s_{\mathcal{L}}$ of s to $\mathcal{L}$, and a unique section $\bar{P} \colon \operatorname{Spec} R_F \to \mathfrak{X}_{R_F}$ extending P. Note that $\mathrm{H}^0(\operatorname{Spec} R_F, \bar{P}^*\mathcal{L})$ is a free rank 1 module over $\operatorname{Spec} R_F$, with a rational section $\bar{P}^* s_{\mathcal{L}} \in$

$\mathrm{H}^0(\bar{P}^*\mathcal{L}) \otimes_{R_F} F \cong F$, so it makes sense to talk of the valuation of the element $\bar{P}^* s_{\mathcal{L}}$ measured with respect to the rank 1 free module $\bar{P}^*\mathcal{L}$. For $u = s(P)$, define

$$v_{\mathcal{L}}(u) := \text{ valuation of the rational section } \bar{P}^* s_{\mathcal{L}} \text{ of } \bar{P}^*\mathcal{L}\,.$$

Then $v_{\mathcal{L}}$ is a $\mathbb{Q}$-valuation on L. See [BG06, Example 2.7.20] for more details.

More generally, we can obtain $\mathbb{Q}$-valuations as follows.

Example 2.4. Let $\|\cdot\|$ be a locally bounded and continuous real-valued metric on L (see [BG06, §2.7]). Then $\log_{\mathbb{R}} \|\cdot\|$ has the scaling property that we want from a valuation. So if

$$v_{\|\cdot\|}(u)) := -\log_{\mathbb{R}} \|u\| \cdot \log_{\mathbb{R}}(p)^{-1} \tag{3}$$

is $\mathbb{Q}$-valued, then it defines a $\mathbb{Q}$-valuation on L.

Definition 2.5. Let $f\colon X' \to X$ be a morphism of smooth proper varieties over $\overline{\mathbb{Q}}_p$. Let v_L be a valuation on L. Then we define the *pullback valuation* f^*v_L on $L' := f^*L$ as follows: For every $x' \in X'$, there is canonical identification of fibers $L'_{x'} \cong L_{f(x')}$, which glue to give the map $\tilde{f}\colon L'^{\times} \to L^{\times}$ in the commutative diagram below.

$$\begin{array}{ccc} L'^{\times} & \xrightarrow{\tilde{f}} & L^{\times} \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \end{array} \tag{4}$$

Then

$$f^*v_L := v_L \circ \tilde{f}\,.$$

Definition 2.6. Let v_L be a valuation on L and let M be another line bundle on X with a valuation v_M.

(1) We define the *sum valuation* on $L \otimes M$ by
$$(v_L + v_M)(u \otimes w) := v_L(u) + v_M(w)\,,$$
where $u \in L_x$ and $w \in M_x$ are nonzero, and $x \in X$.

(2) We call an isomorphism $g\colon L \to M$ an *isometry* (with respect to v_L and v_M) if
$$v_M \circ g = v_L\,.$$

Remark 2.7. The class of $\mathbb{Q}$-valuations is closed under taking sums and pullbacks.

Now let $K/\mathbb{Q}_p$ be a finite extension and let A/K be an abelian variety.

Remark 2.8. Recall the notion of a rigidification of a line bundle from § 1.1. Suppose that L_1 and L_2 are isomorphic line bundles on A with respective rigidifications r_1 and r_2. Then an isomorphism of the rigidified line bundles (L_1, r_1) and (L_2, r_2) is an isomorphism $\varphi\colon L_1 \to L_2$ such that $\varphi(r_1) = r_2$. This isomorphism exists and is unique. Henceforth, we rigidify every line bundle over A by fixing a K-point in the fiber above 0, following [BG06, §9.5].

In particular, if (L, r) is a rigidified symmetric (respectively antisymmetric) line bundle on A, there is a unique isomorphism (15) (respectively (16)) of rigidified line bundles, and by [BG06, Theorem 9.5.4] there is a unique valuation such that this isomorphism is an isometry. This valuation also has the following nice properties.

Proposition 2.9. *(*[BG06, Theorem 9.5.7], [Bet17, Lemma 3.1]*) For every rigidified line bundle (L, r) on A , there is a unique valuation v_L on L with the following properties:*

(a) v_L only depends on (L, r) up to isomorphism of rigidified line bundles.
(b) $v_{L\otimes M} = v_L + v_M$.
(c) $v_{\mathcal{O}_A}(x, a) = \mathrm{ord}_p(a)$ if we choose the rigidification 1 of $\mathcal{O}_A$.
(d) $v_{\varphi^*L} = \varphi^* v_L$ for homomorphisms $\varphi\colon A' \to A$ of abelian varieties over $\mathbb{Q}_p$.
(e) v_L is locally constant on $L^{\times}(K)$.
(f) v_L is $\mathbb{Q}$-valued and has bounded denominator on $L^{\times}(K)$.
(g) $v_L(r) = 0$.

We call v_L the *canonical valuation* associated to (L, r). Betts calls the canonical valuation the *Néron log-metric* in [Bet17]. We have chosen different terminology to avoid confusion with the log functions discussed in the next few sections.

Remark 2.10. It is easy to see that changing the rigidification r to a rigidification r' changes the canonical valuation by the constant $\mathrm{ord}_p(\lambda)$, where $r' = \lambda r$. See [BG06, Remark 9.5.9].

Remark 2.11. In [BG06, §9.5], canonical valuations are constructed using a dynamical approach. See [GK17, Example 8.15] for a construction of canonical valuations based on tropical geometry.

Remark 2.12. If A has good reduction, then canonical valuations are model valuations on the Néron model by [BG06, Example 9.5.22]. In general, canonical valuations are not model valuations; they are not even induced by a formal model, see [Gub03].

3. p-adic Arakelov theory

In this section we recall the parts of Vologodsky (and Coleman) integration theory and p-adic Arakelov theory that will be used in later sections. The main result is Proposition 3.4, which associates a certain p-adic analytic function called a log function to a line bundle equipped with a curvature form. Log functions (Definition 3.3) are the p-adic analytic analogue of the valuations in Definition 2.1, and will be used in Section 5 to define the local contribution at p in the decomposition of the global p-adic height function. Essentially everything we need can be found in Sections 2 and 4 of [Bes05].

3.1. Vologodsky functions.

Coleman's integration theory [Col82, Col85b], originally developed as a theory of iterated integration on certain overconvergent spaces with good reduction over closed subfields of $\mathbb{C}_p$, was recast as a theory of canonical paths in fundamental groupoids of the same spaces in [Bes02], and a theory of Coleman functions is derived from the theory of paths. Shortly afterwards [Vol03], canonical paths for the fundamental groupoid of arbitrary smooth varieties over finite extensions of $\mathbb{Q}_p$ were constructed by Vologodsky, and the associated theory of Vologodsky functions, which we now recall, was derived in [Bes05, Section 2]. Note that Vologodsky functions below are called (Vologodsky) Coleman functions in [Bes05].

Let K be a finite extension of $\mathbb{Q}_p$, with a choice of embedding $K \to \overline{K}$ into an algebraic closure. Let X be a smooth, geometrically connected algebraic variety over K. We first summarize the results in [Bes05]. For this we fix a branch, denoted log, of the p-adic logarithm. We insist that it takes K-values on $K^\times$.

Theorem 3.1. *Let X and* log *be as above.*

(a) *For any locally free sheaf $\mathcal{F}$ on X there is a K-vector space $\mathcal{F}_V(X)$ of Vologodsky functions with values in $\mathcal{F}$, and in particular Vologodsky functions and differential forms, $\mathcal{O}_V(X) = \Omega^0_V(X)$ and $\Omega^i_V(X)$, and differentials $d : \Omega^i_V(X) \to \Omega^{i+1}_V(X)$, such that the sequence*

$$0 \to K \to \mathcal{O}_V(X) \xrightarrow{d} \Omega^1_V(X) \xrightarrow{d} \Omega^2_V(X) \tag{5}$$

is exact. There are products $\Omega^i_V(X) \otimes \Omega^j_V(X) \to \Omega^{i+j}_V(X)$ compatible with the differentials.

(b) *Let $\Omega^i(X)$ be the space of global holomorphic i-forms on X, and let $\Omega^i_{\mathrm{loc}}(X)$ (*[Bes02, Definition 3]*) be the space of locally-analytic $\overline{K}$ valued i-forms on X. There are embeddings*

$$\Omega^i(X) \hookrightarrow \Omega^i_V(X) \hookrightarrow \Omega^i_{\mathrm{loc}}(X) \tag{6}$$

compatible with differentials and products.

(c) *Let z be a coordinate on $\mathbb{A}^1$. There exists a function $\log(z) \in \mathcal{O}_V(\mathbb{A}^1 - \{0\})$ with the property that $d\log(z) = \frac{dz}{z}$ in $\Omega^i(X) \hookrightarrow \Omega^i_V(X)$ and that as a locally analytic function it is our chosen branch of the p-adic logarithm (with values in $\overline{K}$).*

(d) *Let $f : X \to Y$ be a map of varieties. Then there are pullback maps f^* on spaces of Vologodsky functions and differential forms which are compatible with the differentials, the product structure and with the embeddings (6).*

(e) *The induced presheaves $U \to \Omega^i_V(U)$ on the Zariski site of X are sheaves.*

(f) *There is a submodule $\Omega^1_{V,1}(X) \subset \Omega^1_V(X)$ of "one time iterated forms" and a "delbar" operator $\bar\partial : \Omega^1_{V,1}(X) \to \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$ sitting in a short exact sequence*

$$0 \to \Omega^1(X) \to \Omega^1_{V,1} \xrightarrow{\bar\partial} \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)\ , \tag{7}$$

such that when X is affine, the sequence is exact on the right and $\bar\partial(\omega \int \eta) = \omega \otimes [\eta]$ for any two forms ω, η on X, with $[\eta]$ the cohomology class of η.

Proof. This follows from Section 2 of [Bes05]. We point to the relevant references. The product structure and the differential in (a) and the interpretation as locally analytic functions in (b) are all discussed at the bottom of page 321. The exact diagram in (a) is Theorem 2.3. Functoriality as in (d) is discussed at the bottom of page 322. The fact that the integral of $\frac{dz}{z}$ is $\log(z)$, with the "universal" branch of the logarithm is proved in [Vol03, Theorem 1.16(5)]; we get (c) by specializing to the given branch. Returning to [Bes05], the sheaf property (e) is Proposition 2.6 there, and the $\bar{\partial}$ operator is recalled immediately following the proof of this proposition, with (f) being Proposition 2.7. ☐

Remark 3.2.

(i) The space $\mathcal{O}_V(X)$ consists of locally analytic functions that locally look like iterated Vologodsky integrals. We will discuss these integrals in more detail in the special case of curves below.
(ii) The space $\mathcal{O}_{V,1}(X)$ consists of Vologodsky 1-forms that locally look like $\omega \int \eta$ for $\omega, \eta \in \Omega^1(X)$.
(iii) Compared to [Bes05], we have chosen to reverse the order of the terms $\Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$ in (f).
(iv) When X has good reduction, Vologodsky functions essentially coincide with Coleman functions and the resulting theories of iterated integrals are the same in this case. (See [Bes05, Remark 2.13].)
(v) The space $\mathcal{O}_V(\mathrm{Spec}(K))$ is identified with K by taking the value at the underlying physical point. In particular, by functoriality, the values of Vologodsky functions on K-rational points of X are always in K.

3.2. **Log functions for line bundles with curvature forms.** Let L be a line bundle over X.

Definition 3.3. [Bes05, Definition 4.1] A *log function* on L is a function $\log_L \in \mathcal{O}_V(L^\times)$ that satisfies the following two conditions.

- For any $x \in X$, any nonzero u in the fiber L_x and a nonzero constant $\lambda \in \overline{\mathbb{Q}}_p$ one has $\log_L(\lambda u) = \log(\lambda) + \log_L(u)$.
- $d\log_L \in \Omega^1_{V,1}(L^\times)$.

The pair $(L, \log_L)$ will be called a *metrized line bundle* on X, and $\log_L$ a metric on L.

The second condition in the definition of log functions allows one to associate a certain curvature form to log functions, analogous to the construction of metrized line bundles over $\mathbb{R}$. The key result about p-adic log functions is [Bes05, Proposition 4.4], which shows that log functions are determined (up to the addition of the integral of a holomorphic form) by their corresponding curvature forms:

Proposition 3.4. *Suppose that X is proper. Let $\pi\colon L^\times \to X$ be the projection.*

(a) *Suppose that $\overline{L} = (L, \log_L)$ is a metrized line bundle on X such that*
$$\mathrm{ch}_1(L) \in \mathrm{im}\left(\cup : \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X) \to H^2_{\mathrm{dR}}(X)\right).$$
Then there exists a unique element, $\mathrm{Curve}(\overline{L}) \in \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$, *such that* $\pi^* \mathrm{Curve}(\overline{L}) = \bar{\partial} d \log_L$. *The element* $\mathrm{Curve}(\overline{L})$ *is called the* curvature form *of the metrized line bundle* $\overline{L}$ *and it satisfies the relation* $\cup \mathrm{Curve}(\overline{L}) = \mathrm{ch}_1(L)$.
(b) *Conversely, suppose that L is a line bundle on X and that $\alpha \in \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$ satisfies $\cup\alpha = \mathrm{ch}_1(L)$. Then there exists a log function $\log_L$ on L such that the curvature of $(L, \log_L)$ is α.*

If $\overline{L} = (L, \log_L)$, then we sometimes write $\mathrm{Curve}(\log_L)$ for $\mathrm{Curve}(\overline{L})$, and we call $\mathrm{Curve}(\log_L)$ the curvature form of $\log_L$.

Remark 3.5. If $\log_L$ and $\log'_L$ are two different log functions for the same curvature form, then
$$d(\log'_L - \log_L) \in \pi^*\Omega^1(X) \subset \Omega^1(L^\times) \subset \Omega^1_{V,1}(L^\times),$$
since $\ker(\bar{\partial}) = \Omega^1(L^\times)$ by Theorem 3.1 (a), and since the difference of any two log functions is constant along fibers of $\pi\colon L^\times \to X$ by the first defining property of a log function. In the case of curves, this can also be seen from the explicit construction of log functions we give in §3.3.3, where we solve for a meromorphic form with prescribed residues. The space of such meromorphic forms, if non-empty, is a torsor for $\Omega^1(X)$.

Remark 3.6. When X is an abelian variety, then $\mathrm{ch}_1(L) \in \mathrm{im}\left(\cup : \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X) \to H^2_{\mathrm{dR}}(X)\right)$ for every line bundle L. One can explicitly write down curvature forms for the Poincaré bundle (see Proposition 6.11), and this induces curvature forms, and hence log functions on every line bundle on X. (See Definition 4.31.) Moreover, the curvature form determines the log function up to a linear form.

Example 3.7. [Trivial log function on the trivial bundle] For any variety X as above, we have the trivial log function $\log^{\mathrm{triv}}_{\mathcal{O}_X}$ on $\mathcal{O}_X$ with curvature 0 defined by
$$\log^{\mathrm{triv}}_{\mathcal{O}_X}(x, u) := \log(u),$$
where we have used the isomorphism $\mathcal{O}_X^\times \cong X \times \mathbb{G}_m$ and the fixed branch of the p-adic logarithm.

Definition 3.8.

(1) (Tensor products, [Bes05, Definition 4.3]) If $(L, \log_L)$ and $(M, \log_M)$ are two metrized line bundles with corresponding curvature forms α and β, then $\log_L \otimes \log_M$ is a log function for $L \otimes M$ with curvature $\alpha + \beta$.
(2) (Roots of curvatures and log functions) Let L be a line bundle and let $M := L^{\otimes m}$ for some nonzero integer m. If $\log_M$ is a log function for M with curvature form α, then we have an associated log function $\log_L := \frac{1}{m}\log_M$ for L with curvature form α/m defined as follows. Let $s \in L^\times$. Then,
$$\log_L(s) = \frac{1}{m}\log_M(s) := \frac{1}{m}\log_M(s^{\otimes m}).$$
(3) (Pullbacks, [Bes05, Proposition 4.6]) If $(L, \log_L)$ is a metrized line bundle on a smooth, geometrically connected variety Y/K with curvature α and $f\colon X \to Y$ is a morphism, then $(f^*L, f^*(\log_L))$ is a metrized line bundle on X with curvature $f^*\alpha$.
(4) (Isometries) Let $(L, \log_L), (M, \log_M)$ be metrized line bundles on X, with an isomorphism of line bundles $L \cong M$. Let $\tilde{f}\colon L^\times \to M^\times$ be the induced morphism. We say that $(L, \log_L)$ and $(M, \log_M)$ are *isometric* if $\tilde{f}^*(\log_M) = \log_L$.

3.3. **The case of curves.** We now provide more details on iterated integrals and log functions in the one-dimensional case. This is all we need for our application to Quadratic Chabauty in Section 7. In particular, we sketch how to explicitly construct a log function starting with a given curvature form.

3.3.1. *Iterated integrals as Vologodsky functions.* Suppose that X is one-dimensional. As there are no locally analytic 2-forms on X we have $\Omega^2_{\mathrm{loc}}(X) = 0$ and therefore the differential $d : \mathcal{O}_V(X) \to \Omega^1_V(X)$ is surjective. If $\omega_1, \dots, \omega_k \in \Omega^1(X)$, then we can iteratively define
$$\int_x^z \omega_1 \circ \cdots \circ \omega_k = \int_x^z \left(\omega_1 \int_x^z \omega_2 \circ \cdots \circ \omega_k\right),$$
where $\int_x^z$ means the unique preimage under d which vanishes at x. We can view this iterated integral as the v_0-component of a solution of the differential equation
$$dv_k = 0,\ dv_{k-1} = \omega_k v_k,\ \dots, dv_0 = \omega_1 v_1$$
with $v_k = 1$, or, in a different language, as the v_0-component of the horizontal section for the connection
$$\nabla(v) = dv - v \cdot A, \text{ with } A = \begin{pmatrix} 0 & \omega_1 & 0 & \cdots & 0 \\ 0 & 0 & \omega_2 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & \cdots & \omega_k \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{8}$$

3.3.2. *Single and double integrals on curves.* Let X/K be a nice curve. Let $x \in X$ and let z be a coordinate in a Zariski neighborhood of x. By abuse of notation, we also let $\log(z)$ denote the Vologodsky function defined on a punctured neighborhood of x obtained by pulling back the function in Theorem 3.1 (c) by the coordinate z and using Theorem 3.1 (d). We also suppress the choice of base point used to normalize iterated integrals below, and assume that both sides are normalized correctly so that equality holds.

Lemma 3.9. *Let η be a meromorphic form on X and let $\omega \in \Omega^1(X)$. Let $c_x := \mathrm{Res}_x(\eta)$. Then, there is an open neighborhood V_x of x in the p-adic topology, and a Laurent series g centered at x such that, with respect to a local parameter z at x on V_x, we have on $V_x \setminus \{x\}$ the equality $\int^z \eta = g(z) + c_x \log(z)$. Furthermore, if $c_x = 0$, then $\mathrm{Res}_x(g\omega)$ is independent of the choice of g, and in this case, $\int^z(\omega \int^z \eta) = h(z) + \mathrm{Res}_x(g\omega)\log(z)$ on $V_x \setminus \{x\}$.*

Proof. For two parameters z and z' as above the function $\log(\frac{z}{z'})$ is analytic near x, which shows that the validity of the lemma is independent of the choice of z. Thus we may assume that z is an algebraic uniformizer at X, defined on a Zariski open neighborhood U_x of x with no poles of η and no poles or zeros of z except possibly at x. Let $\eta := \sum_{i \geq \mathrm{ord}_x(\eta)} a_i z^i dz$ be the Laurent expansion of η on V_x centered at x, so that $a_{-1} = c_x$. Let $\eta' := \sum_{i<-1} a_i z^i dz$ and $\eta'' := \eta - \eta' - c_x \frac{dz}{z}$. Observe that $\eta' = df$ with $f(z) := \sum_{i<-1} \frac{a_i z^{i+1}}{i+1} \in \mathcal{O}_V(U_x \setminus \{x\})$, and so we have $\int \eta' = f$ (up to adjusting f by a constant) on $V_x \setminus \{x\}$. Also, note that $\eta'' \in \Omega^1_V(U_x)$, so we have $\int \eta'' \in \mathcal{O}_V(U_x)$ and this has a power series expansion in z on some analytic neighborhood V_x of x. Setting $g := \int \eta'' + f$, we see that on $V_x \setminus \{x\}$, the function g has a Laurent series expansion $g(z)$ and
$$\int^z \eta = \int^z \eta'' + \int^z \eta' + \int^z c_x \frac{dz}{z} = g(z) + c_x \log(z).$$

Since $dg = \eta - c_x \frac{dz}{z}$, and any two choices of g satisfying this differ by a constant, $\mathrm{Res}_x(g\omega)$ is independent of the choice of g. When $c_x = 0$, we have $g\omega - \mathrm{Res}_x(g\omega)\frac{dz}{z} \in \Omega^1_{\mathrm{loc}}(V_x \setminus \{x\})$, and by writing down an explicit Laurent series expansion with no residue and arguing as before, we can find a Laurent series h centered at x such that on $V_x \setminus \{x\}$ we have $dh = g\omega - \mathrm{Res}_x(g\omega)\frac{dz}{z}$, and hence

$$\int^z (\omega \int^z \eta) = \int^z g\omega = \int^z dh + \int^z \mathrm{Res}_x(g\omega)\frac{dz}{z} = h(z) + \mathrm{Res}_x(g\omega)\log(z). \qquad \square$$

Remark 3.10. Note that locally around any point $x \in X$, iterated Coleman integrals are also defined and are polynomials in $\log(z)$ with coefficients which are Laurent series [Bes05, Section 5]. One can alternatively prove the lemma by instead using the local comparison between Vologodsky iterated integrals and Coleman iterated integrals. For the first case of the lemma [BZ21] suffices, whereas for the second part one also needs [KL]

Example 3.11. Let X be a smooth, geometrically integral curve, not necessarily proper.

(a) For any $f \in K(X)$, invertible on an open subset U, using Theorem 3.1 (a,c,d), it follows that there exists a function $\log(f) \in \mathcal{O}_V(U)$, unique up to an additive constant, such that $d\log(f) = df/f$ and such that it is equal to $\log \circ f$ as a locally analytic function.
(b) Let $\omega \in \Omega^1(X)$ and let η be a form of second kind on X, holomorphic on an open subset U of X. From Lemma 3.9, it follows that there are well-defined functions $f, g \in \mathcal{O}_V(U)$ such that $dg = \eta, df = g\omega$, such that g admits a Laurent series expansion around points in $X \setminus U$, and such that f has an expansion as the sum of a Laurent series and a constant multiple of $\log(z)$ around points in $X \setminus U$. Furthermore, in this case, Theorem 3.1 (f) implies $\bar{\partial} df = \omega \otimes [\eta] \in \Omega^1(U) \otimes H^1_{\mathrm{dR}}(U)$.

3.3.3. *Log functions for curves.* Granting the existence of curvature forms for log functions, we first prove the following lemma that will be useful in the sketch of construction of log functions on curves that follows.

Lemma 3.12. *Assume that $(L, \log_L)$ is a metrized line bundle on a nice curve X/K. Let s be a section of L, invertible on a Zariski open subset U. Then, $\log_L(s) \in \mathcal{O}_V(U)$ and the form $d\log_L(s) \in \Omega^1_V(U)$ admits a locally meromorphic extension to all points $x \in X$, with at worst simple poles and such that $\mathrm{Res}_x(d\log_L(s)) = \mathrm{ord}_x(s)$ for every $x \in X$.*

Proof. We have $\log_L(s) \in \mathcal{O}_V(U)$ as the pullback of $\log_L$ via $s\colon U \to L^\times$. In particular, it is locally analytic on U in consistence with the lemma. Let $x \in X \setminus U$, and let z be a coordinate in a neighborhood of x. Let $n = \mathrm{ord}_x(s)$. Then $z^{-n}s$ is invertible in a Zariski open neighborhood V_x of x, so in turn $\log_L(z^{-n}s) \in \mathcal{O}_V(V_x)$ and $\gamma_x := d\log_L(z^{-n}s) \in \Omega^1_V(V_x)$. Now by the second property of the log function, on $V_x \setminus \{x\}$, we have

$$\log_L(s) = n\log(z) + \log_L(z^{-n}s), \tag{9}$$

and hence by definition of $\log(z)$,

$$d\log_L(s) = n\frac{dz}{z} + \gamma_x.$$

Since the γ_x are locally analytic on V_x, the expression on the right hand side gives a locally meromorphic extension of $d\log_L(s)$ to points $x \in X \setminus U$, as a form with a simple pole of residue $\mathrm{ord}_x(s)$. $\square$

Sketch of construction of log functions for curves. We briefly sketch a construction of a log function with a given curvature α when X is a curve, assuming the existence of log functions, since this is what we will need for the application to Quadratic Chabauty. In the case of curves, the problem reduces to solving for a meromorphic differential γ with prescribed polar parts, as we explain below. We refer the reader to the proof of [Bes05, Proposition 4.4] which shows that log functions exist in any dimension using a careful argument using Cěch cocycles.

Let $\alpha = \sum_i \omega_i \otimes [\eta_i] \in \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$ for some holomorphic forms $\omega_i \in \Omega^1(X)$ and forms of second kind η_i with corresponding cohomology classes $[\eta_i] \in H^1_{\mathrm{dR}}(X)$, such that $\cup\alpha = \mathrm{ch}_1(L)$. We know by Proposition 3.4 that a log function on L with curvature α exists and is unique up to the integral of a holomorphic form on X. Pick a nonzero meromorphic section s of L, invertible on a Zariski open subset U of X as in Lemma 3.12. As we will see later, a log function on L is completely determined by $\log_L(s)$, so to construct one log function with curvature α (hence all), it suffices to determine $\log_L(s)$ up to the integral of a holomorphic form. We now restrict further to some $U' \subset U$ where all the η_i are holomorphic. Since $\bar{\partial} d\log_L(s) = \alpha|_{U'}$, and we know $\ker(\bar{\partial})$ from Theorem 3.1 (f). Example 3.11 (b) shows that on X, we have

$$d\log_L(s) = \sum \omega_i \int \eta_i + \gamma, \tag{10}$$

for some meromorphic form γ on X. Since $\log_L(u)$ satisfies Lemma 3.12, the polar parts of γ at points in U are exactly the negative of those of $\sum \omega_i \int \eta_i$ and the same is true at $x \in X \setminus U$ – except that there is a difference in the logarithmic part, which is determined by the condition (that automatically holds also at $x \in U$),

$$\operatorname{Res}_x(\gamma) = \operatorname{ord}_x(s) - \operatorname{Res}_x\left(\sum \omega_i \int \eta_i\right).$$

Because γ is a meromorphic form and therefore satisfies the Residue Theorem, a necessary condition of the existence of γ is that $\sum_{x\in X} \operatorname{Res}_x(\sum \omega_i \int \eta_i) = \sum_{x\in X} \operatorname{ord}_x(s)$, and by Riemann-Roch this is also a sufficient condition. It is easy to see, independently of the general theory of log functions, that this condition is indeed satisfied because

$$\sum_{x\in X} \operatorname{Res}_x(\sum \omega_i \int \eta_i) = \cup\alpha = \operatorname{ch}_1(L)$$

by assumption and we know that $\sum_{x\in X} \operatorname{ord}_x(s) = \operatorname{ch}_1(L)$. As the conditions above completely determine the polar parts of γ at every point, the degree of freedom of γ is exactly $\Omega^1(X)$, and therefore determining γ with the right polar conditions is equivalent to determining $\log_L$.

It remains to show how $\log_L$ is determined by $\log_L(s) \in \mathcal{O}_V(U)$. By the very definition of log functions (Definition 3.3) it clearly determines $\log_L$ above U so it suffices to extend $\log_L$ to $x \in X \setminus U$. For such an x let z be a local coordinate as in the proof of Lemma 3.12. Then (9) gives $\log_L(z^{-n}s) = \log_L(s) - n\log(z)$. As $\log_L$ satisfies the conditions of Lemma 3.12, the function $\log_L(s) - n\log(z)$ is analytic in an analytic neighborhood of x. Hence it extends to x and we can set $\log_L(z^{-n}s)(x) = (\log_L(s) - n\log(z))(x)$. It is easy to see that this extension is independent of the choice of z. □

Example 3.13. [Log functions for tangent bundles on hyperelliptic curves, [BBM16]] Let $X\colon y^2 = f(x)$ be an odd degree hyperelliptic curve In [BBM16], an explicit log function for the tangent bundle of X is constructed, extending the elliptic case treated in [BB12]. This is then applied to express a suitably normalized local Coleman-Gross height pairing in terms of double integrals, which is one of the main ingredients for the Quadratic Chabauty methods for integral points on $X$when $f \in \mathbb{Z}[x]$ is monic and $\operatorname{rk}(\operatorname{Jac}(X)/\mathbb{Q}) = g(X)$.

We recall the main features; for details see the proof of [BBM16, Theorem 2.2]. Let $\{\omega_i := x^i dx/2y\}_{i=0}^{2g-1}$ be the standard basis for $H^1_{\mathrm{dR}}(X)$, and let $\{\overline{\omega_i}\}_{i=0}^{g-1}$ be a basis for a complementary subspace W to $\Omega^1(X) \subset H^1_{\mathrm{dR}}(X)$ dual to the standard basis $\{\omega_i\}_{i=0}^{g-1}$ of $\Omega^1(X)$ under the cup product pairing. We choose the curvature form $\alpha := -2\sum_{i=0}^{g-1} \omega_i \otimes [\overline{\omega_i}]$ for the tangent bundle $\mathcal{T}$. Fix the section θ of the tangent bundle dual to the holomorphic form ω_0; this section has a pole of order $2g-2$ at the unique point at ∞ and no other zeroes or poles. Since both $d\log_{\mathcal{T}}(\theta)$ and the form $-2\sum_{i=0}^{g-1} \omega_i \int \overline{\omega_i}$ have at worst simple poles at the unique point at ∞ and have the same residue at ∞, the meromorphic differential γ we need to solve for is in fact holomorphic on X. It follows that for any $\omega \in \Omega^1(X)$,

$$\log_{\mathcal{T}}(\theta) = -2\sum_{i=0}^{g-1} \omega_i \int \overline{\omega_i} + \int \omega,$$

is a log function for $\mathcal{T}$ with curvature α, and that every log function for $\mathcal{T}$ with curvature α is of this form for some $\omega \in \Omega^1(X)$.

4. Canonical log functions on line bundles on abelian varieties

Let A be an abelian variety over a p-adic field K. The goal of this section is a construction of canonical log functions on A, similar to the canonical valuations in Proposition 2.9. Instead of the limit constructions typically applied in the real-valued setting, we will use the curvature forms of metrized line bundles introduced in Proposition 3.4.

This idea is not new; for instance, Moret-Bailly has used curvature forms to characterize canonical real-valued metrics on abelian varieties over $\mathbb{C}$ in [MB85]. However, in that setting the curvature form determines the metric up to a constant, and the canonical metric is characterized by having a translation-invariant curvature form in $H^{1,1}(A_{\mathbb{C}})$. In our setting, curvature forms are valued in $\Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$ and hence automatically translation-invariant, for instance by [Bar57]. Furthermore, the curvature form does not uniquely determine the log function: If L is a line bundle on A, and $\alpha \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$ is such that $\cup\alpha = \operatorname{ch}_1(L)$, then there are several possible log function $\log_L$ with $\operatorname{Curve}(\log_L) = \alpha$, one for each holomorphic form on A. We will use this degree of freedom in Theorem 4.13 to show that when L is symmetric, there exists a unique "good" log function (see Definition 4.9), on L with curvature α. This is not true in the antisymmetric case, which is more involved, see Theorem 4.18. To define canonical good log functions, we show that there is a choice of curvature form α for the Poincaré bundle on A so that the good log function with curvature α induces good log functions on all symmetric and antisymmetric line bundles on A by pullback and restriction. The so obtained log functions will be our p-adic analytic analogue of the canonical valuation. It will also serve as the component at p of a canonical p-adic adelic metrized line bundle when A is defined over a number field, see Section 5.

4.1. **Geometry of abelian varieties.** We recall the theory of line bundles on abelian varieties, the dual abelian variety and the Poincaré line bundle. In this subsection A is an abelian variety over an arbitrary field K of characteristic 0. We write s, d, π_i for the addition and subtraction maps and the projections $A \times A \to A$. Let L be a line bundle on A.

Definition 4.1. We call L symmetric (respectively antisymmetric) if there exists an isomorphism $(-1)^*L \cong L$ (respectively $(-1)^*L \cong L^{-1}$.

Remark 4.2. Pullbacks of symmetric line bundles by morphisms of abelian varieties are also symmetric, and the tensor product of two symmetric line bundles on an abelian variety is also symmetric.

The following results are well-known.

Proposition 4.3.

(a) If L is symmetric, then there is an isomorphism of line bundles on $A \times A$

$$s^*L \otimes d^*L \cong (\pi_1^*L)^2 \otimes (\pi_2^*L)^2 \ . \tag{11}$$

(b) The line bundle L is antisymmetric if and only if the cohomology class of L (in de Rham cohomology) is trivial. In this case we have an isomorphism of line bundles on $A \times A$:

$$s^*L \cong \pi_1^*L \otimes \pi_2^*L \ . \tag{12}$$

Proof. See [Lan83, Proposition 5.2.4] for (a) and [Lan83, Proposition 5.2.3] for Equation (12). It is clear that if a line bundle is antisymmetric, then its cohomology class is 0 because -1 acts as 1 on $H^2_{\mathrm{dR}}(A)$. Conversely, we may use the Lefschetz principle to reduce to the case of abelian varieties over $\mathbb{C}$ and replace de Rham cohomology with Betti cohomology. It then follows from the theorem of Appel-Humbert ([Mum08, Page 19]) that cohomologically trivial line bundles correspond to elements of $\mathrm{Hom}(\mathrm{H}_1(A,\mathbb{Z}), \mathbb{S}^1)$, were $\mathbb{S}^1$ is the unit circle inside $\mathbb{C}^\times$, and these are all clearly antisymmetric, which proves (b). □

Corollary 4.4. *If L is symmetric (respectively antisymmetric), then for any morphisms $f, g \colon X \to A$ of K-varieties, one has*

$$(f+g)^*L \otimes (f-g)^*L \cong (f^*L)^2 \otimes (g^*L)^2 \tag{13}$$

(respectively

$$(f+g)^*L \cong f^*L \otimes g^*L) \, . \tag{14}$$

In particular, setting $f = g = \mathrm{id}$ or directly pulling back via $\Delta : A \to A \times A$, we get isomorphisms

$$[2]^*L \cong L^{\otimes 4} \tag{15}$$

when L is symmetric and

$$[2]^*L \cong L^{\otimes 2} \tag{16}$$

when L is antisymmetric.

We now recall the theory of the dual abelian variety and the Poincaré line bundle. For any variety X, the Picard variety of X represents the relative Picard functor

$$S \mapsto \mathrm{Pic}_{X/S} := \mathrm{Pic}(X \times S)/\,\mathrm{Pic}(S)$$

where Pic is the usual Picard functor of isomorphism classes of line bundles and $\mathrm{Pic}(S)$ maps to $\mathrm{Pic}(X \times S)$ by pullback. The connected component of the Picard variety classifies the subfunctor mapping S to the isomorphism classes of line bundles which are fiber by fiber antisymmetric. In the special case of an abelian variety we denote this connected component by $\hat{A}$ - the *dual abelian variety.* Universality provides a line bundle $\mathcal{P}$ on $A \times \hat{A}$ whose restriction to $A \times \{0\}$ is trivial. We normalize $\mathcal{P}$ up to isomorphism by insisting that its restriction to $\{0\} \times \hat{A}$ is also trivial.

Definition 4.5. We call $\mathcal{P}$ the *Poincaré line bundle* on $A \times \hat{A}$.

Lemma 4.6. *The Poincaré line bundle is symmetric.*

Proof. See [BG06, Theorem 8.8.4]. □

Proposition 4.7. *Let L/A be a line bundle.*

(a) The line bundle

$$\phi_L(a) := t_a^* L \otimes L^{-1}$$

is antisymmetric.

(b) The map ϕ_L is induced from a morphism $\phi_L\colon A \to \hat{A}$ of abelian varieties.

(c) If L is ample, then ϕ_L is an isogeny.

(d) The bundle L is antisymmetric if and only if ϕ_L is trivial.

(e) If L is symmetric, then there exists an isomorphism $L^{\otimes 2} \cong (\mathrm{id} \times \phi_L)^* \mathcal{P}$, with $\mathrm{id} \times \phi_L : A \to A \times \hat{A}$.

Proof. For the first two assertions, see [BG06, Theorem 8.5.1]. The third one is [BG06, Proposition 8.5.5] and for the fourth see [BG06, Theorem 8.8.3]. We give a short proof of (e). Let L be symmetric and let $\mathrm{id} \times \phi_L : A \times A \to A \times \hat{A}$. From the definition of ϕ_L, it follows that

$$(\mathrm{id} \times \phi_L)^* \mathcal{P} \cong s^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L'$$

for some line bundle L' on A. By restricting to $\{0\} \times A$ it is easy to see that $L' \cong L^{-1}$. By symmetry of L the right hand side in (e) becomes

$$\Delta^*(s^* L \otimes \pi_1^* L^{-1} \otimes \pi_2^* L^{-1}) \cong [2]^* L \otimes L^{\otimes -2} \cong L^{\otimes 2}\,.$$

□

4.2. **Good log functions on symmetric line bundles.** For the remainder of this section we suppose that K is a p-adic field and that A/K is an abelian variety. In the following, all line bundles on abelian varieties will be rigidified at 0. All (iso)morphisms of line bundles will be viewed as (iso)morphisms of rigidified line bundles, although we will often not write this explicitly, to simplify notation. Likewise, tensor products of line bundles will be tensor products of rigidified line bundles. For a symmetric or antisymmetric line bundles, the rigidification fixes the choice of isomorphism (11), (12), (13), (14), (15) and (16)

Definition 4.8. We say that a log function $\log_L$ on a rigidified line bundle (L, r) on A/K is *normalized* if $\log_L(r) = 0$.

We first study the analogue of canonical valuations in the case of symmetric line bundles.

Definition 4.9. We say that a log function on a symmetric line bundle L/A is *good* if (11) is an isometry.

Remark 4.10. A good log function on a symmetric line bundle is normalized.

Lemma 4.11. *Let L/A be a symmetric line bundle and let $\log_L$ be a log function on L.*

(a) There is a holomorphic form $\omega \in \Omega^1(A)$ such that the difference of the induced log function on $L^{\otimes 4}$ and the pullback of the log function on $[2]^*L$ by the isomorphism (15) is the integral of 2ω.

(b) Assume further that the isomorphism $[-1]^*(L) \cong L$ is an isometry. Then for the holomorphic form $\omega \in \Omega^1(A)$ in part (a), we also have that the difference of the induced log functions on the two sides of the isomorphism (11) is the integral of $\pi_1^*\omega + \pi_2^*\omega$. Hence $\log_L$ is good if and only if $\omega = 0$.

Proof. Let α be the curvature of $(L, \log_L)$. Since $s^* = \pi_1^* + \pi_2^*$ and $d^* = \pi_1^* - \pi_2^*$ as maps $H^1_{\mathrm{dR}}(A) \to H^1_{\mathrm{dR}}(A \times A)$, a direct calculation shows that $s^*(\omega' \otimes [\eta]) + d^*(\omega' \otimes [\eta]) = 2\pi_1^*(\omega' \otimes [\eta]) + 2\pi_2^*(\omega' \otimes [\eta])$ for all $\omega' \otimes [\eta] \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$ and therefore

$$s^*\alpha + d^*\alpha = 2\pi_1^*\alpha + 2\pi_2^*\alpha.$$

Therefore the two sides of the isomorphism (11) have the same curvature. Further pullback by the diagonal map $A \to A \times A$ shows that the two sides of the isomorphism (15) also have the same curvature.

(a) This follows from Remark 3.5.

(b) Since $\Omega_1(A \times A) = \pi_1^*\Omega_1(A) \oplus \pi_2^*\Omega_2(A)$, by Remark 3.5, there is a holomorphic form $\pi_1^*\omega_1 + \pi_2^*\omega_2$ such that the induced log functions differ by the integral of $\pi_1^*\omega_1 + \pi_2^*\omega_2$. Since L is symmetric, and we have $[-1]^*(\log_L) = \log_L$ by assumption, the induced log functions on both sides of (11) are invariant with respect to switching the two coordinates on A. From this, it follows that their difference is also invariant with respect to switching the two coordinates, and hence $\omega_1 = \omega_2 = \omega$. The behavior with respect to (15) follows by further pullback along the diagonal $A \to A \times A$. □

Lemma 4.12. *A log function on a symmetric line bundle L/A is good if and only if (15) is an isometry. Furthermore, for a good log function, the map (13) is also an isometry for any choices of morphisms $f, g\colon X \to A$ of K-varieties.*

Proof. Note that the isomorphism (11) with the induced log functions being an isometry implies that the isomorphism (15) is also an isometry by further pullback along the diagonal. This proves one direction of the first statement.

Let $\log_L$ be a log function such that the isomorphism (15) is an isometry. In order to show that $\log_L$ is good, by Lemma 4.11(b), it now suffices to show that the pullback of $\log_{[-1]^*L}$ by the isomorphism $\iota\colon L \to [-1]^*(L)$ is equal to $\log_L$. It is easy to see that a log function on L with a given curvature for which (15) is an isometry is unique (see the proof of Theorem 4.13). Therefore, it suffices to show that $\iota^*\log_{[-1]^*L}$ also makes the unique isomorphism of rigidified bundles $[2]^*(L) \cong L^{\otimes 4}$ in (15) an isometry. Since $[-1]^*$ commutes with $[2]^*$ and taking tensor powers, it follows that the induced isomorphism $\kappa\colon [2]^*([-1]^*L) \cong ([-1]^*L)^{\otimes 4}$ obtained by further pullback of the isometry $[2]^*(L) \cong L^{\otimes 4}$ (with log functions on the two sides induced from $\log_L$) by $[-1]$ is also an isometry. Consider the isomorphism $\iota^{\otimes -4}\cdot\kappa\cdot[2]^*(\iota)\colon [2]^*(L) \cong L^{\otimes 4}$. This composition is an isometry with the log functions on the two sides induced from the log function $\iota^*\log_{[-1]^*L}$ on L by taking pullbacks and tensor powers.

For the good log function, the map (13) is also an isometry for any choices of morphisms $f, g\colon X \to A$, since isometries are preserved by further pullbacks. □

Theorem 4.13. *Let L be a symmetric line bundle on A and let $\alpha \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$ such that $\cup\alpha = \mathrm{ch}_1(L)$. Then there exists a unique good log function $\log_L$ such that $\mathrm{Curve}(\log_L) = \alpha$. The good log function satisfies $[-1]^*\log_L = \log_L$.*

Proof. Pick some log function $\log'_L$ on L with curvature form α. We will also let $\log'_{[2]^*L}$ denote the induced log function on $L^{\otimes 4}$ under the unique isomorphism of rigidified bundles $[2]^*(L) \cong L^{\otimes 4}$ in (15). By Lemma 4.11(a), there is an $\omega \in \Omega_1(A)$ such that

$$\log'_{L^{\otimes 4}} - \log'_{[2]^*L} = 2\int \omega. \tag{17}$$

Let $\log_L := \log'_L - \int\omega$. A direct computation shows that the log functions on $[2]^*(L)$ and $L^{\otimes 4}$ induced by $\log_L$ are $\log'_{[2]^*L} - 2\int\omega$ and $\log'_{L^{\otimes 4}} - 4\int\omega$, respectively. Combined with Equation (17), this shows that the isomorphism (15) is an isometry with the induced log functions from $\log_L$. Furthermore, by Remark 3.5, it follows that $\log_L$ is the only log function with this property. We are now done by Lemma 4.12. □

We now prove a result that can be used to show that certain isomorphisms between symmetric line bundles are isometries.

Lemma 4.14. *Let L/A be a symmetric line bundle and let $\log_L$ be a good log function with respect to the chosen rigidification r.*

(a) Let $f\colon A' \to A$ be a homomorphism of abelian varieties. Then $f^*\log_L$ is good.
(b) Let M/A be another symmetric line bundle with a good log function $\log_M$. Then $\log_L + \log_M$ is good.
(c) Let $\varphi\colon L_1 \to L_2$ be an isomorphism between line bundles on abelian variety A'/K such that L_1 and L_2 are obtained from L by pullbacks and tensor products as in (a) and (b). For $i = 1, 2$, let $\log_i$ be the log function on L_i induced by $\log_L$ and let $\alpha_i = \mathrm{Curve}(\log_i)$. Then φ is an isometry if and only if $\alpha_1 = \alpha_2$.

Proof. As f is a homomorphism, pullback by f followed by pullback by s, d, π_1, π_2 is the same as first pulling back by s, d, π_1, π_2 followed by pullback by $f \times f$. Using Remark 4.2, the first two assertions are immediate. For $i = 1, 2$, there is a unique good log function on L_i with curvature α_1. By (a) and (b), $\log_1$ and $\log_2$ are good, so (c) follows. □

4.3. **Good and canonical log functions on antisymmetric line bundles.**

Definition 4.15. We say that a log function on an antisymmetric line bundle L is *good* if (12) is an isometry.

As in the symmstric case, the notion of goodness depends on the choice of a rigidification of L, used to fix the isomorphism (12).

Remark 4.16. A good log function on an antisymmetric line bundle is normalized.

Definition 4.17. We call a metrized line bundle *flat* if the corresponding curvature form is trivial.

As the cohomology class $\mathrm{ch}_1(L)$ of any antisymmetric line bundle L/A is trivial, any $\alpha \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$ cupping to 0 is the curvature form for some log function on L.

Theorem 4.18. *A log function $\log_L$ on an antisymmetric line bundle L is good if and only if $(L, \log_L)$ is flat and normalized.*

Proof. By Remark 4.16, a good log function is normalized. If (12) is an isometry, so is (16) by further pullback along the diagonal map. We show that if (16) is an isometry with the induced log functions, then $\alpha := \mathrm{Curve}(\log_L) = 0$. Since

$$s^*(\omega' \otimes [\eta]) = (\pi_1^* + \pi_2^*)(\omega' \otimes [\eta]) = \pi_1^*(\omega' \otimes [\eta]) + \pi_2^*(\omega' \otimes [\eta]) + \pi_1^*\omega' \otimes \pi_2^*[\eta] + \pi_2^*\omega' \otimes \pi_1^*[\eta],$$

for all $\omega' \otimes [\eta] \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(p)$, it follows that if $\Delta\colon A \to A \times A$ is the diagonal map, then $[2]^*\alpha = \Delta^* s^*\alpha = 4\alpha$. We also have $\alpha^{\otimes 2} = 2\alpha$. Since the log functions and hence the curvatures of the two sides of (16) agree by assumption, it follows that $4\alpha = 2\alpha$ and hence $\alpha = 0$.

It now suffices to show that any log function on an antisymmetric line bundle with curvature 0 is good. To prove this, we will construct an explicit good, in particular, normalized, log function $\log_L$ with curvature 0 in Proposition 4.25. By Remark 3.5, any other log function with curvature 0 is of the form $\log'_L := \log_L + \int \omega$ for some holomorphic form $\omega \in \Omega^1(A)$, where $\int \omega$ is normalized to have the value 0 at 0. Once again, since $s^*\omega = \pi_1^*\omega + \pi_2^*\omega$, replacing $\log_L$ by $\log'_L$, by Theorem 3.1(d), adds $\pi_1^* \int \omega + \pi_2^* \int \omega$ to the induced log functions on both sides of (12), and therefore $\log'_L$ is also good. ☐

Remark 4.19. Alternatively, one can construct a good log function on L by writing L as $t_a^*M \otimes M^{-1}$ for some symmetric ample line bundle M and $a \in A$ and showing that, if we start with any good log function on M, the induced log function on $t_a^*M \otimes M^{-1}$ is good.

Corollary 4.20. *Let L be an antisymmetric line bundle. A log function on L is good if and only if* (16) *is an isometry. Furthermore, for a good log function on L, the map* (14) *is also an isometry for any choices of morphisms $f, g\colon X \to A$ of K-varieties.*

Proof. The first statement follows from Theorem 4.18. Isometries are preserved by pullback. ☐

Remark 4.21. Observe that for symmetric line bundles, there is a *unique* good log function for *any* curvature form for L, whereas for antisymmetric line bundles, good log functions exist *only* if the curvature is 0. Moreover, in this case, *every* normalized log function with curvature 0 is automatically good, so good log functions are far from unique for antisymmetric line bundles.

We can metrize an antisymmetric line bundle L/A by restricting a log function on the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ to the fiber corresponding to L. Since $\mathcal{P}$ is symmetric, we can fix a curvature form for $\mathcal{P}$ and obtain, by Theorem 4.13, a unique associated good log function $\log_{\mathcal{P}}$. We will show that for certain curvature forms on $\mathcal{P}$, this yields a good log function on L. This construction completes the proof of Theorem 4.18 and simultaneously attaches a unique good log function to every antisymmetric line bundle.

For notational convenience we denote A by A_1 and $\hat{A}$ by A_2 and we write π_i for the projection $A_1 \times A_2 \to A_i$. There are direct sum decompositions

$$\Omega^1(A \times \hat{A}) = \pi_1^*\Omega^1(A) \oplus \pi_2^*\Omega^1(\hat{A}) \;, H^1_{\mathrm{dR}}(A \times \hat{A}) = \pi_1^* H^1_{\mathrm{dR}}(A) \oplus \pi_2^* H^1_{\mathrm{dR}}(\hat{A})$$

resulting in a direct sum decomposition

$$\Omega^1(A \times \hat{A}) \otimes H^1_{\mathrm{dR}}(A \times \hat{A}) = \oplus_{i,j=1,2} H_{ij} \;, H_{ij} = \pi_i^*\Omega^1(A_i) \otimes \pi_j^* H^1_{\mathrm{dR}}(A_j) \cong \Omega^1(A_i) \otimes H^1_{\mathrm{dR}}(A_j).$$

For an element $\alpha \in \Omega^1(A \times \hat{A}) \otimes H^1_{\mathrm{dR}}(A \times \hat{A})$ we let α_{ij} be its component in H_{ij}.

Definition 4.22. An element $\alpha \in \Omega^1(A \times \hat{A}) \otimes H^1_{\mathrm{dR}}(A \times \hat{A})$ is called a *purely mixed curvature form* for the Poincaré bundle $\mathcal{P}$ if $\cup\alpha = \mathrm{ch}_1(\mathcal{P})$ and $\alpha_{00} = \alpha_{11} = 0$.

We will show in Section 6.2 that purely mixed curvature forms for the Poincaré bundle exist. In Proposition 6.11, we prove that purely mixed curvature forms for $\mathcal{P}$ are in one-to-one correspondence with complementary subspaces for the inclusion $\Omega^1(A) \hookrightarrow H^1_{\mathrm{dR}}(A)$. First we give an alternative characterization of the good log function on $\mathcal{P}$ with a given purely mixed curvature form.

Remark 4.23. From now on, we fix a rigidification $r_{\mathcal{P}}$ of $\mathcal{P}$ at $(0, 0)$. This choice induces trivializations $\mathcal{P}_{\{0_A\}\times\hat{A}} \cong \mathcal{O}_{\hat{A}}$ and $\mathcal{P}_{A\times\{0_{\hat{A}}\}} \cong \mathcal{O}_A$ by requiring that $r_{\mathcal{P}}$ corresponds to $1(0_A)$ (respectively $1(0_{\hat{A}})$), where 1 is the canonical section of $\mathcal{O}_A$ (respectively of $\mathcal{O}_{\hat{A}}$). In particular, for $\hat{a} \in \hat{A}$, this choice induces a rigidification $r_{\hat{a}}$ on the antisymmetric line bundle $\mathcal{P}_{A\times\{\hat{a}\}}$ on A, corresponding to $1(\hat{a})$, and similarly for $a \in A$.

Proposition 4.24. *Let α be a purely mixed curvature form for $\mathcal{P}$.*

(a) *There is a unique log function $\log_{\mathcal{P}}$ on $\mathcal{P}$ with curvature α which restricts to the trivial log function on $A \times \{0\}$ and on $\{0\} \times \hat{A}$ with respect to the trivializations from Remark 4.23.*

(b) *The log function* $\log_{\mathcal{P}}$ *from part* (a) *is the good log function with curvature* α *with respect to the rigidification* $r_{\mathcal{P}}$ *from Remark 4.23.*

Proof.

(a) Since α is purely mixed, the restriction of α to any horizontal or vertical fiber is trivial. Thus, by Theorem 4.18, any log function for α induces flat log functions on each antisymmetric line bundle on A and by duality a flat log function on each antisymmetric line bundle on $\hat{A}$. In particular, the restriction of such a log function to $A \times \{0\}$ and $\{0\} \times \hat{A}$ is trivial up to an integral of a holomorphic form. By Remark 3.5, the curvature determines the log function up to the integral of a holomorphic one form $\omega = \pi_1^*\omega_1 + \pi_2^*\omega_2 \in \Omega^1(A \times \hat{A})$, for some $\omega_1 \in \Omega^1(A), \omega_2 \in \Omega_1(\hat{A})$. Changing the log function by the integral of ω changes the log function restricted to $A \times \{0\}$ by $\int \omega_1$ and the restriction to $\{0\} \times \hat{A}$ by $\int \omega_2$. From this (a) follows easily.

(b) By Corollary 4.12 it is enough to prove that (15) is an isometry with the log functions on the two sides induced from $\log_{\mathcal{P}}$ from part (a). In other words, it is enough to show that the log function $\log'_{\mathcal{P}}$ on $\mathcal{P}$ given by $1/4\,([2]^* \log_{\mathcal{P}})$ (see Definition 3.8 (2)) equals $\log_{\mathcal{P}}$. For this, first observe that $\log'_{\mathcal{P}}$ and $\log_{|p}$ have the same curvature, since $[2]^*$ acts as multiplication by 2 on $H^1_{\mathrm{dR}}(A)$ and $\Omega^1(A)$. Since $\log'_{\mathcal{P}}$ and $\log_{\mathcal{P}}$ both restrict to the trivial log function on $\{0\} \times \hat{A}$ and $A \times \{0\}$, we are done by part (a). □

Proposition 4.25. *Let* α *be a purely mixed curvature form for* $\mathcal{P}$ *and let* $\hat{a} \in \hat{A}$. *Then the log function induced by* $\log_{\mathcal{P}}$ *on* $\mathcal{P}|_{A\times\{\hat{a}\}}$ *is good for the rigidification* $r_{\hat{a}}$ *from Remark 4.23.*

Proof. Consider the maps

$$\tilde{s} = s \times \mathrm{id},\ \tilde{\pi}_1 = \pi_1 \times \mathrm{id},\ \tilde{\pi}_2 = \pi_2 \times \mathrm{id} \colon A \times A \times \hat{A} \to A \times \hat{A}.$$

We claim that there exists an isomorphism

$$\tilde{s}^*\mathcal{P} \cong \tilde{\pi}_1^*\mathcal{P} \otimes \tilde{\pi}_2^*\mathcal{P}\,. \tag{18}$$

Indeed, since the restriction of $\mathcal{P}$ to any fiber is antisymmetric, by (12) for every $\hat{a} \in \hat{A}$ we have

$$s^*(\mathcal{P}|_{A\times\{\hat{a}\}}) \cong \pi_1^*(\mathcal{P}|_{A\times\{\hat{a}\}}) \otimes \pi_2^*(\mathcal{P}|_{A\times\{\hat{a}\}})\,, \tag{19}$$

and hence $\tilde{s}^*\mathcal{P}|_{A\times A\times\{\hat{a}\}} \cong \tilde{\pi}_1^*\mathcal{P}_{A\times A\times\{\hat{a}\}} \otimes \tilde{\pi}_2^*\mathcal{P}_{A\times A\times\{\hat{a}\}}$. By the seesaw principle, the two sides of (18) are isomorphic up to the pullback of a line bundle L on $\hat{A}$. But then restricting to $\{0\} \times \{0\} \times \hat{A}$ we immediately see that $L \cong \mathcal{O}_{\hat{A}}$, which implies (18). We normalize the isomorphism in (18) by requiring that it is an isomorphism of rigidified line bundles, with rigidifications induced by Remark 4.23.

Next we claim that $\tilde{s}^* \log_{\mathcal{P}}$ and $\tilde{\pi}_1^* \log_{\mathcal{P}} \otimes \tilde{\pi}_2^* \log_{\mathcal{P}}$ have the same curvature. Because α is purely mixed, it suffices to show that $\tilde{s}^*(\omega \otimes [\eta]) = \tilde{\pi}_1^*(\omega \otimes [\eta]) \otimes \tilde{\pi}_2^*(\omega \otimes [\eta])$ for $\omega \in \Omega^1(A)$ and $[\eta] \in H^1_{\mathrm{dR}}(\hat{A})$ and for $\omega \in \Omega^1(\hat{A})$ and $[\eta] \in H^1_{\mathrm{dR}}(A)$. Indeed, since $s^* = \pi_1^* + \pi_2^*$ on $\Omega^1(A)$

$$\tilde{s}^*(\omega \otimes [\eta]) = (\pi_1^*\omega + \pi_2^*\omega) \otimes [\eta] = \tilde{\pi}_1^*(\omega \otimes [\eta]) + \tilde{\pi}_2^*(\omega \otimes [\eta]).$$

But since $s^* = \pi_1^* + \pi_2^*$ on $H^1_{\mathrm{dR}}(A)$, we also get the analogous statement for $\omega \in \Omega^1(\hat{A})$ and $[\eta] \in H^1_{\mathrm{dR}}(A)$. Therefore, by Lemma 4.6 and Lemma 4.14, we have

$$\tilde{s}^* \log_{\mathcal{P}} = \tilde{\pi}_1^* \log_{\mathcal{P}} + \tilde{\pi}_2^* \log_{\mathcal{P}}\,,$$

which shows that the isomorphism (18) is an isometry. Finally, it is not hard to check that the restricted isomorphism (19) sends the pullback of $r_{\hat{a}}$ along s to the sum of the pullbacks of $r_{\hat{a}}$ along π_1 and π_2. The claim follows. □

If α is a purely mixed curvature form for $\mathcal{P}$ and if L/A is an antisymmetric line bundle with rigidification r, then there is a unique isomorphism $\psi_{L,r} \colon (L, r) \cong (\mathcal{P}|_{A\times\{\hat{a}\}}, r_{\hat{a}})$, where $\hat{a}$ is the class of L.

Definition 4.26. Let L/A be an antisymmetric line bundle with rigidification r. The *canonical log function on* L (with respect to α) is the log function obtained by pulling back the log function from Proposition 4.25 by $\psi_{L,r}$.

We immediately deduce:

Corollary 4.27. *Let* $\log_L$ *be the canonical log function on an antisymmetric rigidified line bundle* (L, r). *Then* $\log_L$ *is good. In particular, it is normalized.*

We now show that the canonical log function on an antisymmetric line bundle has some desirable properties. First, by dualizing the objects in the proof of Proposition 4.25, we get the following result.

Proposition 4.28. *The tensor product of the canonical log functions on two antisymmetric line bundles is canonical.*

Proof. Let $\hat{a}_1, \hat{a}_2 \in \hat{A}$. If we swap the roles of A and $\hat{A}$ in the proof of Proposition 4.25, we find that

$$\tilde{s}^*\mathcal{P}|_{\{\hat{a}_1\}\times\{\hat{a}_2\}\times A} \cong \tilde{\pi}_1^*\mathcal{P}|_{\{\hat{a}_1\}\times\{\hat{a}_2\}\times A} \otimes \tilde{\pi}_2^*\mathcal{P}|_{\{\hat{a}_1\}\times\{\hat{a}_2\}\times A}\,. \tag{20}$$

is an isometry for the log functions induced by the good log function on $\mathcal{P}$ with respect to α, where

$$\tilde{s} = s \times \mathrm{id},\ \tilde{\pi}_1 = \pi_1 \times \mathrm{id},\ \tilde{\pi}_2 = \pi_2 \times \mathrm{id} \colon \hat{A} \times \hat{A} \times A \to \hat{A} \times A.$$

By duality, the result follows. □

To understand the behavior of canonical log functions on antisymmetric line bundles under translations, we first prove a preliminary result.

Lemma 4.29. *Let $(L, \log_L), (M, \log_M)$ be metrized line bundles on $A \times \hat{A}$, with an isomorphism of line bundles $L \cong M$. Suppose $\mathrm{Curve}(\log_L) = \mathrm{Curve}(\log_M)$, and that the restrictions of $\log_L$ and $\log_M$ to $A \times \{0\}$ and $\{0\} \times \hat{A}$ are isometric. Then the given isomorphism is an isometry.*

Proof. Write log_M, by abuse of notation, for the pulled back log function on L By Remark 3.5, the curvature determines the log function up to the integral of a holomorphic one form $\omega \in \Omega^1(A \times \hat{A})$, so we have

$$\log_L = \log_M + \pi_1^* \int \omega_1 + \pi_2^* \int \omega_2$$

for some $\omega_1 \in \Omega^1(A), \omega_2 \in \Omega_1(\hat{A})$. Restricting both sides of the equality above to $A \times \{0\}$ and $\{0\} \times \hat{A}$ and using our assumption that the restrictions of the log functions to the fibers are isometric, it follows that $\int \omega_1 = \int \omega_2 = 0$, and therefore $\log_L = \log_M$. □

Proposition 4.30. *Let L/A be an antisymmetric line bundle with canonical log function $\log_L$ and let $a \in A$. Then, up to an additive constant, $t_a^* \log_L$ is the canonical log function on $t_a^* L$.*

Proof. Let $\tau = t_{(a,0)} \colon A \times \hat{A} \to A \times \hat{A}$. Then, for $\hat{a} \in \hat{A}$, since $\mathcal{P}|_{A\times\{\hat{a}\}}$ is translation-invariant, we have

$$\tau^*\mathcal{P}|_{A\times\{\hat{a}\}} \cong \mathcal{P}|_{A\times\{\hat{a}\}} \cong (\mathcal{P} \otimes \pi_2^* M_a)|_{A\times\{\hat{a}\}},$$

where M_a is the line bundle on $\hat{A}$ whose class is a under duality. Moreover, the restriction of $\tau^*\mathcal{P}$ to $\{0\} \times \hat{A}$ is isomorphic to $\pi_2^* M_a$, so $\tau^*\mathcal{P}|_{\{0\}\times\hat{A}} \cong (\mathcal{P} \otimes \pi_2^* M_a)|_{\{0\}\times\hat{A}}$, which implies

$$\tau^*\mathcal{P} \cong \mathcal{P} \otimes \pi_2^* M_a\,. \tag{21}$$

Let $\log_{\mathcal{P}}$ be the good log function on $\mathcal{P}$ (with respect to a purely mixed curvature form α and the rigidification $r_{\mathcal{P}}$, as usual) and let $\log_{M_a}$ be the canonical log function on M_a induced by $\log_{\mathcal{P}}$ via restriction. We want to use Lemma 4.29 to show that (21) is an isometry for these log functions. First note that $\mathrm{Curve}(\log_{\mathcal{P}})$ is translation-invariant and that $\pi_2^* \log_{M_a}$ has trivial curvature. Furthermore, (21) is tautologically an isometry when restricted to $\{0\} \times \hat{A}$. It is also an isometry when restricted to $A \times \{0\}$, by Remark 4.24 and since the trivial bundle equipped with the trivial log function is translation-invariant. Hence (21) is indeed an isometry as claimed.

Finally, let $\hat{a} = [L] \in \mathrm{Pic}^0(A)$. Restricting the isometry (21) to the fiber $A \times \{\hat{a}\}$, the result follows, since $\pi_2^* \log_{M_a}|_{A\times\{\hat{a}\}}$ is constant. □

4.4. **Canonical log functions on arbitrary line bundles.** Fix a purely mixed curvature form α for $\mathcal{P}$. Let $\log_{\mathcal{P}}$ be the good log function with curvature α from Proposition 4.24. Now we systematically fix canonical (with respect to α) log functions on all line bundles on A using $\log_{\mathcal{P}}$. We have already defined canonical log functions for antisymmetric line bundles in Definition 4.26.

For a line bundle L/A, we set $L^+ := L \otimes [-1]^* L$ and $L^- := L \otimes ([-1]^* L)^{-1}$, so that L^+ is symmetric, L^- is antisymmetric and we have

$$L^{\otimes 2} \cong L^+ \otimes L^-. \tag{22}$$

Recall the homomorphism ϕ_L from Proposition 4.7.

Definition 4.31. Let L be a line bundle on A with rigidification r. The *canonical log function* $\log_L$ for L (with respect to α and r) is defined as follows:

(a) When L is antisymmetric, then $\log_L$ is defined in Definition 4.26.
(b) When L is symmetric, then $\log_L$ is the good log function with curvature $\frac{1}{2}(\mathrm{id} \times \phi_L)^* \alpha$ (with respect to r).

(c) In general, we define $\log_L = \frac{1}{2}(\log_{L^+} + \log_{L^-})$ using the canonical decomposition (22) and Definition 3.8 (2), where the log functions on L^+ and L^- are the canonical log functions with respect to the rigidifications r^+ (respectively r^-) on L^+ (respectively L^-) induced by r.

Remark 4.32. Note that the definition in part (c) above is compatible with parts (a) and (b) by Proposition 4.28 and Proposition 4.14 (b), since $L^+ \cong L^{\otimes 2}$ and $L^- \cong \mathcal{O}_A$ for L symmetric, and analogously $L^+ \cong \mathcal{O}_A$ and $L^- = L^{\otimes 2}$ for L antisymmetric.

Letting $\hat{a} = [L^-] \in \hat{A}$, we get the following formula for the canonical log function for (L, r) with respect to α as a sum of (scaled) pullbacks:

$$\log_L = \frac{1}{2}\psi^*_{L^+,r^+}(\mathrm{id} \times \phi_L)^* \log_{\mathcal{P}} + \frac{1}{2}\psi^*_{L^-,r^-} \log_{\mathcal{P}} |_{A\times\{\hat{a}\}}\,, \tag{23}$$

where $\psi_{L^+,r^+} \colon L^+ \cong L^+$ is the unique isomorphism sending r^+ to $(\mathrm{id} \times \phi_L)^* r_{\mathcal{P}}$.

Remark 4.33. By construction and by Remarks 4.10 and 4.16, the canonical log function is normalized. In analogy with Remark 2.10, changing the rigidification r to a rigidification $r' = \lambda r$ changes the canonical log function by $\log \lambda$.

5. p-adic adelic valuations and global p-adic heights

Shou-Wu Zhang has introduced a theory of real-valued adelic metrics on line bundles on varieties over number fields in [Zha95]. Such an adelic metric is a family of continuous real-valued metrics, one for every place of the number field. Using the previous two sections, we now develop a p-adic version of this theory, a theory of adelic line bundles on nice varieties over number fields with values in $\mathbb{Q}_p$. Via a choice of an idelé class character, we can use this to develop a fairly general theory of p-adic heights. We then specialize to the case of abelian varieties and canonical p-adic heights, similar to Zhang's construction of Néron-Tate heights using adelic metrics (see also [CL11] and [BG06, §9.5] for expositions).

Let K be a number field. For a place v of K, we denote by K_v the completion of K at v with ring of integers $\mathcal{O}_v$ and uniformizer π_v We also fix an algebraic closure $\bar{K}_v$ of K_v.

Let p be a prime number and let

$$\chi = \sum_v \chi_v \colon \mathbb{A}_K^\times / K^\times \to \mathbb{Q}_p\,.$$

be a continuous idelé class character. This means that χ is a continuous homomorphism such that

- we have $\chi_{\mathfrak{q}}(\mathcal{O}_{\mathfrak{q}}^\times) = 0$ for $\mathfrak{q} \nmid p$;
- for every $\mathfrak{p} \mid p$, there is a $\mathbb{Q}_p$-linear trace map $t_{\mathfrak{p}}$ such that we can decompose

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{p}}^\times & \xrightarrow{\ \chi_{\mathfrak{p}}\ } & \mathbb{Q}_p. \\ & {\scriptstyle \log_{\mathfrak{p}}} \searrow \quad \nearrow {\scriptstyle t_{\mathfrak{p}}} & \\ & K_{\mathfrak{p}} & \end{array} \tag{24}$$

One can think of χ as a "global log". We shall assume that χ is ramified at all $\mathfrak{p} \mid p$, so that we can extend (24) to $K_{\mathfrak{p}}$, leading to a factorization

$$\chi_{\mathfrak{p}} = t_{\mathfrak{p}} \circ \log_{\mathfrak{p}}\,, \tag{25}$$

valid on $K_{\mathfrak{p}}^\times$. Then $\log_{\mathfrak{p}}$ is a branch of the logarithm $K_{\mathfrak{p}}^\times \to K_{\mathfrak{p}}$ as in Section 3 and 4.

Suppose that X is a nice variety over K. If v is a finite place of X, then we write X_v for the base change of X to K_v and $\bar{X}_v$ for the base change to $\bar{K}_v$. Similarly, for a line bundle L of X, we write L_v and $\bar{L}_v$.

5.1. p-adic adelic metrics and heights.

Definition 5.1. Let L be a line bundle on X/K. A *p-adic adelic metric* on L consists of the following data:

- for every place $\mathfrak{p} \mid p$, a log function $\log_{L,\mathfrak{p}}$ on $L_{\mathfrak{p}}$ (see Definition 3.3),
- for every finite place $\mathfrak{q} \nmid p$, a $\mathbb{Q}$-valuation $v_{L,\mathfrak{q}}$ on $L_{\mathfrak{q}}$ (see Definition 2.1),

satisfying the following compatibility condition: There exists an integral model $\mathcal{X}/\mathcal{O}_K$ of X and an extension $\mathcal{L}$ of L to $\mathcal{X}$ such that for all but finitely many $\mathfrak{q} \nmid p$, the valuation $v_{L,\mathfrak{q}}$ is equal to the model valuation $v_{\mathcal{L}\otimes\mathcal{O}_{\mathfrak{q}}}$ (see Example 2.3).

We call $\overline{L} := (L, (\log_{L,\mathfrak{p}})_{\{\mathfrak{p}\mid p\}}, (v_{L,\mathfrak{q}})_{\{\mathfrak{q}\nmid p\}})$ a *p-adic adelically metrized line bundle* on X.

There are obvious notions of tensor products and pullbacks of adelic metrics and adelically metrized line bundles via the corresponding notions in Sections 2 and 3.

Definition 5.2. Let $\overline{L} = (L, (\log_{L,\mathfrak{p}})_{\{\mathfrak{p}|p\}}, (v_{L,\mathfrak{q}})_{\{\mathfrak{q}\nmid p\}})$ be an adelically metrized line bundle on X/K such that for every place $\mathfrak{p} \mid p$, the branch of the logarithm in Definition 3.3 is the branch $\log_{\mathfrak{p}}$ induced by $\chi_{\mathfrak{p}}$ as in (25). The *p-adic height* associated with $\overline{L}$ and χ is the function $h_{\overline{L}} := h_{\overline{L},\chi}$ associating with a point $x \in X(K)$ the value

$$h_{\overline{L}}(x) = \sum_{\mathfrak{p}|p} t_{\mathfrak{p}}(\log_{L,\mathfrak{p}}(u)) + \sum_{\mathfrak{q}\nmid p} v_{L,\mathfrak{q}}(u)\chi_{\mathfrak{q}}(\pi_{\mathfrak{q}}) \in \mathbb{Q}_p\,,$$

where $u \in L_x(K) \setminus \{0\}$.

Note that $\log_{L,\mathfrak{p}}(u) \in K_{\mathfrak{p}}$ by (v) of Remark 3.2, so that the formula makes sense. The height is independent of the choice of u by the fact that χ is an idelé class character.

Remark 5.3. It would be interesting to generalize our construction of the p-adic height from points to subvarieties of X.

5.2. **Canonical p-adic heights on abelian varieties.** Let A/K be an abelian variety. As in previous sections, we equip all line bundles L/A with a rigidification r at 0, inducing compatible rigidifications at 0 on all L_v.

Definition 5.4. Let L/A be a line bundle. An adelic metric on L is called *good* if the log function at $\mathfrak{p}$ is good for all $\mathfrak{p} \mid p$ and if the valuations at all $\mathfrak{q} \nmid p$ are canonical.

Proposition 5.5. *Let $\overline{L}$ be a line bundle on A endowed with a good adelic metric. Then the p-adic height $h_{\overline{L}}$ is a quadratic function on $A(K)$. It is a quadratic (respectively linear) form if L is symmetric (respectively antisymmetric).*

Proof. This follows at once from the definitions. □

Recall that for all $\mathfrak{q} \nmid p$, there is a unique good $\mathbb{Q}$-valuation on the line bundle $L_{\mathfrak{q}}$ with respect to a rigidification r. To define canonical adelic metrics, we fix a purely mixed curvature form α_p of the Poincaré bundle $\mathcal{P}_{\mathfrak{p}} := \mathcal{P} \otimes K_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$.

Definition 5.6. Let L/A be a line bundle. For each $\mathfrak{q} \nmid p$, let $v_{L,\mathfrak{q}}$ be the canonical valuation and for each $\mathfrak{p} \mid p$ let $\log_{L,\mathfrak{p}}$ be the canonical log function. Then we call $((\log_{L,\mathfrak{p}})_{\{\mathfrak{p}|p\}}, (v_{L,\mathfrak{q}})_{\{\mathfrak{q}\nmid p\}}))$ the *canonical adelic metric* on L.

Lemma 5.7. *The canonical adelic metric is good.*

Proof. This follows from Definition 4.31 and Corollary 4.27. □

Let $\overline{L} := (L, (\log_{L,\mathfrak{p}})_{\{\mathfrak{p}|p\}}, (v_{L,\mathfrak{q}})_{\{\mathfrak{q}\nmid p\}}))$ be a canonical adelic metrized line bundle. We denote the height function $h_{\overline{L}} = h_{\overline{L},\chi}$ by $\hat{h}_L$. Then, as a special case, we obtain the canonical p-adic height pairing

$$\hat{h} := \hat{h}_{\mathcal{P}} \colon A(K) \times \hat{A}(K) \to \mathbb{Q}_p \tag{26}$$

on $A(K) \times \hat{A}(K)$. It depends both on $\underline{\alpha}$ and on χ, but not on the rigidification of $\mathcal{P}$ by Remarks 2.10 and 4.33.

Corollary 5.8. *The canonical height pairing is bilinear.*

Proof. This follows at once from Proposition 5.5. Alternatively, note that the adelic metric on $\mathcal{P}$ restricts over $\{a\} \times \hat{A}$ and over $A \times \{a'\}$ to good adelic metrics on antisymmetric line bundles by Proposition 4.24. The induced heights are thus linear by Proposition 5.5, which implies bilinearity. □

Corollary 5.9. *Let L be a line bundle on A and let $\hat{a} = [L^-] \in \hat{A}$ be the point corresponding to $L^- = L \otimes ([-1]^*L)^{-1}$.*

(a) For all $a \in A(K)$, we have

$$\hat{h}_L(a) = \frac{1}{2}\hat{h}(a, \phi_L(a) + \hat{a})$$

(b) The bilinear form associated with $\hat{h}_L$ is

$$(a, b) \mapsto \frac{1}{4}\left(\hat{h}(a, \phi_L(b)) + \hat{h}(b, \phi_L(a))\right).$$

Proof. More precisely, the local contributions at all places to both sides in (a) are equal. For $v \nmid p$, this follows at once from Proposition 2.9. For $\mathfrak{p} \mid p$, it is implied by the construction of the canonical log function and by (23). Part (b) is obvious. □

6. Comparison with Mazur-Tate and Coleman-Gross heights

In this section, we first show that the canonical heights constructed in Section 5 induce height pairings in the sense of Mazur and Tate [MT83]. We then restrict to the Jacobian of the curve. In this case, the canonical height pairing with respect to a theta divisor is the same as the height pairing due to Coleman and Gross [CG89], with appropriate choices. By [Bes04, Bes17], this pairing is the same as the geometric pairing constructed by Nekovář [Nek93], if the curve has semistable reduction at all places above p. In future work we will prove a more general result dealing with arbitrary abelian varieties and the Zarhin height pairing [Zar90].

In this section, K denotes a number field and A/K an abelian variety. We choose a continuous idelé class character $\chi\colon \mathbb{A}_K/K^* \to \mathbb{Q}_p$. For $\mathfrak{p} \mid p$, let $\log_\mathfrak{p}$ be the branch of the logarithm induced by $\chi_\mathfrak{p}$ as in (24).

6.1. **Mazur-Tate.** In [MT83], Mazur and Tate construct global height pairings on A using biextensions of A and $\hat{A}$ by $\mathbb{G}_m$. These global pairings are sums of local pairings defined in [MT83, Section 2]. For all non-archimedean places v of K we now define local pairings $\langle\cdot,\cdot\rangle_v$ via canonical valuations. We then show that they satisfy the conditions of [MT83, §2.2]. In other words, the pairing at a place v is induced from a χ_v-splitting in the sense of [MT83, Section 1].

6.1.1. *Local height pairings away from p.* First let $\mathfrak{q}$ be a non-archimedean place of K such that $\mathfrak{q} \nmid p$. Then we define a pairing between zero cycles $\mathfrak{a} = \sum_x n_x x \in Z_0^0(A_{K_\mathfrak{q}})$ of degree 0 and a divisors $D \in \mathrm{Div}(A_{K_\mathfrak{q}})$ with disjoint support as follows: Let s be a meromorphic section of a line bundle L on $A_{K_\mathfrak{q}}$ such that $D = \mathrm{div}(s)$ and let r be a rigidification of L. Let $v_{L,\mathfrak{q}}$ be the canonical valuation on L (with respect to r). We set

$$\langle \mathfrak{a}, D\rangle_\mathfrak{q} := \sum_x n_x v_{L,\mathfrak{q}}(s(x))\chi_\mathfrak{q}(\pi_\mathfrak{q}), \tag{27}$$

where $\pi_\mathfrak{q}$ is a uniformizer at $\mathfrak{q}$. Since $\deg(\mathfrak{a}) = 0$, this is independent of the choice of r.

Lemma 6.1. *The pairing (27) satisfies the following properties:*

(a) $\langle\cdot,\cdot\rangle_\mathfrak{q}$ is biadditive,
(b) $\langle\cdot,\cdot\rangle_\mathfrak{q}$ is translation-invariant,
(c) if $D = \mathrm{div}(f)$, then $\langle \mathfrak{a}, \mathrm{div}(f)\rangle_\mathfrak{q} = \chi_\mathfrak{q}(f(\mathfrak{a}))$

Proof. The pairing (27) is the same, up to a constant, as the real-valued pairing defined by

$$\langle \mathfrak{a}, D\rangle_{\mathfrak{q},\mathbb{R}} := \sum_x n_x v_{L,\mathfrak{q}}(s(x)) \log_\mathbb{R} \mathrm{Nm}(\mathfrak{q}). \tag{28}$$

The pairing (28) satisfies (a)–(c) by the proof of [BG06, Theorem 9.5.11]. ☐

Remark 6.2. The pairing (28) is the classical local Néron symbol at $\mathfrak{q}$, see [BG06, Theorem 9.5.11]. It is characterized uniquely by (a)–(c) and by local boundedness of the function $x \mapsto \langle D, (x) - (b)\rangle_\mathfrak{q}$ for every fixed $b \in A(\overline{K_\mathfrak{q}}) \setminus \mathrm{supp}(D)$. By [MT83, Proposition 2.3.1], the pairing (28) is equal to the canonical $\chi_\mathfrak{q}$-pairing defined in [MT83, §2.3] with respect to the canonical $\chi_\mathfrak{q}$-splitting from [MT83, Theorem 1.5].

6.1.2. *Local height pairings above p.* Let $\mathfrak{p}$ be a prime above p. Let $\mathrm{Div}_a(A_{K_\mathfrak{p}})$ be the subgroup of $\mathrm{Div}(A_{K_\mathfrak{p}})$ consisting of divisors algebraically equivalent to 0. We fix a curvature form $\alpha_\mathfrak{p}$ on $\mathcal{P}_\mathfrak{p}$ for every $\mathfrak{p} \mid p$. Let $\mathfrak{a} = \sum_x n_x x \in Z_0^0(A_{K_\mathfrak{p}})$ be a zero-cycle of degree 0 and let $D \in \mathrm{Div}_a(A_{K_\mathfrak{p}})$ have disjoint support from $\mathfrak{a}$. Suppose that s a meromorphic section of a line bundle L on $A_{K_\mathfrak{p}}$ such that $D = \mathrm{div}(s)$ and let $\log_{L,\mathfrak{p}}$ be the canonical log-function on L induced by $\alpha_\mathfrak{p}$ with respect to an arbitrary choice of rigidification. Then we define

$$\langle \mathfrak{a}, D\rangle_\mathfrak{p} := \sum_x n_x \log_{L,\mathfrak{p}}(s(x)). \tag{29}$$

Lemma 6.3. *The pairing (29) satisfies*

(a) $\langle\cdot,\cdot\rangle_\mathfrak{p}$ is biadditive,
(b) $\langle\cdot,\cdot\rangle_\mathfrak{p}$ is translation-invariant,
(c) if $D = \mathrm{div}(f)$ is principal, then $\langle \mathfrak{a}, D\rangle_\mathfrak{p} = \log_\mathfrak{p} f(\mathfrak{a})$.

In particular, it satisfies the conditions of [MT83, §2.2].

Proof. The first property follows from Proposition 4.28. Now let $\mathfrak{a}$, D and L be as above and let $a \in A$. By Proposition 4.30, the canonical log function $\log_{t_a^*L,\mathfrak{p}}$ on t_a^*L is the same as $\log_{L,\mathfrak{p}}$ up to an additive constant. Since $\deg(\mathfrak{a}) = 0$, we obtain

$$\langle t_a^*(\mathfrak{a}), t_a^*(D)\rangle_{\mathfrak{p}} = \log_{t_a^*L,\mathfrak{p}}(t_a^*(s(\mathfrak{a}))) = \log_{L,\mathfrak{p}}((t_a)_* \circ t_a^*(s(\mathfrak{a}))) = \log_{L,\mathfrak{p}}(s(\mathfrak{a})) = \langle \mathfrak{a}, D\rangle_{\mathfrak{p}} \tag{30}$$

as in the proof of [BG06, Theorem 9.5.11].

For (c), let $D = \mathrm{div}(f)$ be principal and consider f as a section of $\mathcal{O}_A$. Let $\log'$ be the canonical log function on $\mathcal{O}_A$. Then we have $\langle \mathfrak{a}, D\rangle_{\mathfrak{p}} = \log'(f(\mathfrak{a}))$. But by Proposition 4.24, $\log'$ is the trivial log function. □

Remark 6.4. By [MT83, §2.2], the pairing $\langle \cdot, \cdot, \rangle_{\mathfrak{p}}$ is induced from a $\chi_{\mathfrak{p}}$-splitting in the sense of [MT83, Section 1]. Hence the choice of curvature form $\alpha_{\mathfrak{p}}$ induces a χ_p-splitting.

Remark 6.5. In contrast to Lemma 6.1, we cannot hope for the pairing to be uniquely determined by these properties, since the canonical log function depends on the choice $\alpha_{\mathfrak{p}}$.

6.1.3. *Global height pairings.* Let

$$\hat{h} = h_{\mathcal{P},\underline{\alpha},\chi} \colon A(K) \times \hat{A}(K) \to \mathbb{Q}_p$$

be the canonical height pairing (26) relative to the choices $\underline{\alpha}$ and χ. It is explained in [MT83, (3.1.1)] that the sum of the local pairings $\langle \cdot, \cdot\rangle_v$ defines another global pairing

$$\langle \cdot, \cdot\rangle \colon A(K) \times \hat{A}(K) \to \mathbb{Q}_p\,.$$

We now review the construction of the latter pairing and show that both pairings are equal.

For $\mathfrak{a} \in Z_0^0(A)$ and $D \in \mathrm{Div}_a(A)$ with disjoint support, we define

$$\langle \mathfrak{a}, D\rangle := \sum_v \langle \mathfrak{a}, D\rangle_v\,.$$

Lemma 6.6. *The pairing $\langle \cdot, \cdot\rangle$ induces a well-defined pairing on $A(K) \times \hat{A}(K)$.*

Proof. Let $\mathfrak{a} \in Z_0^0(A)$ and $D \in \mathrm{Div}_a(A)$ have disjoint support. It is obvious that $\langle \mathfrak{a}, D\rangle$ does not change if we replace D by another divisor D' such that $\mathcal{O}(D) \simeq \mathcal{O}(D')$. Let $S \colon Z_0^0(A) \to A$ be the summation map; its kernel is generated by cycles of the form $t_x^*Z - Z$. Since D is antisymmetric, $t_{-x}^*D - D = \mathrm{div}(f)$ is principal. We find that

$$\langle t_z^*Z - Z, D\rangle_v = \langle Z, t_{-x}^*D - D\rangle_v = \chi_v(f(Z)) \tag{31}$$

for all v, and all Z and x such that the arguments have disjoint support. Hence $\sum_v \langle \cdot, \cdot\rangle_v$ factors through $Z_0^0(A)/\ker(S) \times \mathrm{Pic}^0(A) = A \times \hat{A}$. □

Corollary 6.7. *Fix a continuous idelé class character $\chi \colon \mathbb{A}_K/K^* \to \mathbb{Q}_p$ and, for every $\mathfrak{p} \mid p$, a purely mixed curvature form $\alpha_{\mathfrak{p}}$ for $\mathcal{P}_{\mathfrak{p}}$. Let $\hat{h} = \hat{h}_{\mathcal{P},\chi,(\alpha_{\mathfrak{p}})_{\mathfrak{p}|p}}$ denote the corresponding p-adic height. Let $\langle \cdot, \cdot\rangle$ denote the Mazur-Tate height pairing with respect to the following χ-splitting: For $\mathfrak{q} \nmid p$, the splitting is the canonical $\chi_{\mathfrak{q}}$-splitting, and for $\mathfrak{p} \mid p$ it is the $\chi_{\mathfrak{p}}$-splitting induced by $\alpha_{\mathfrak{p}}$ as in Remark 6.4.*

Then we have

$$\hat{h}(a, \hat{a}) = \langle a, \hat{a}\rangle$$

for every $a \in A(K)$ and $\hat{a} \in \hat{A}(K)$.

Proof. The definitions of $\hat{h}$ and $\langle \cdot, \cdot\rangle$ imply that

$$\hat{h}(a, \hat{a}) = \hat{h}(a, \hat{a}) - \hat{h}(0, \hat{a}) = \langle (a) - (0), D\rangle,$$

for any D such that $[\mathcal{O}(D)] = \hat{a}$ and such that $a, 0 \notin \mathrm{supp}(D)$. □

6.1.4. *Jacobians.* Let J be the Jacobian of a nice curve X/K. We assume for simplicity that there is an Abel-Jacobi map $\iota\colon X \to J$ defined over K. Let Θ denote the theta divisor with respect to ι. Then J is self-dual via the principal polarization $\phi_\Theta := \phi_{\mathcal{O}(\Theta)}\colon J \to \hat{J}$. More precisely, by [BG06, Proposition 8.10.20], we have

$$(\mathrm{id} \times \phi_\Theta)^*\mathcal{P} \cong s^*\Theta \otimes (\pi_1^*\Theta)^{-1} \otimes (\pi_2^*\Theta)^{-1} =: \delta$$

and $\Delta^*\delta \cong \mathcal{O}(\Theta) \otimes [-1]^*\mathcal{O}(\Theta)$. Therefore, for all $a \in A(K)$, we have

$$h_{\mathcal{P}}(a, \phi_\Theta(a)) = \hat{h}_\delta(a, a) = \hat{h}_{\mathcal{O}(\Theta)\otimes[-1]^*(\mathcal{O}(\Theta))}(a)$$

where the heights are defined with respect to χ and the curvatures induced by $\underline{\alpha}$. We obtain a bilinear pairing

$$\begin{aligned} J(K) \times J(K) &\to \mathbb{Q}_p \\ (a, b) &\mapsto \hat{h}(a, \phi_\Theta(b)) \end{aligned}$$

We can express this global height pairing as a sum of local pairings on the curve. Let v be a non-archimedean place of K and let $z, w \in \mathrm{Div}^0(X_{K_v})$ with disjoint support. Write $w = \mathrm{div}(s)$, where s is a meromorphic section of a line bundle $L \in \mathrm{Pic}^0(X)$. Let $M \in \mathrm{Pic}^0(J)$ such that $L = \iota^*M$ and let $z = \sum_x n_x(x)$. If $v = \mathfrak{q} \nmid p$, then we define

$$\langle z, w\rangle_{\mathfrak{q}} := \sum_x n_x v_{L,\mathfrak{q}}(s(x))\chi_{\mathfrak{q}}(\mathfrak{q}), \tag{32}$$

where $v_{L,\mathfrak{q}} = \iota^* v_{M,\mathfrak{q}}$ and $v_{M,\mathfrak{q}}$ is the canonical valuation on M. Then $\langle \cdot, \cdot \rangle_{\mathfrak{q}}$ is the local Néron symbol on the curve, see [BG06, Theorem 9.5.17].

If $v = \mathfrak{p} \mid p$, let $\log_{M,\mathfrak{p}}$ be the canonical log function on M and let $\log_{L,\mathfrak{p}} = \iota^* \log_{M,\mathfrak{p}}$. Then we define

$$\langle z, w\rangle_{\mathfrak{p}} := \sum_x n_x \log_{L,\mathfrak{p}}(s(x))\,. \tag{33}$$

Write $w = w_1 - w_2$, where both w_i are non-special, and set $b_i = \iota(w_i) \in J$. Then, by [Lan83, Theorem 5.5.8], we have $w_i = \iota^*\Theta^-_{b_i}$, where $\Theta^-_{b_i} = t^*_{b_i}\Theta^-$ and $\Theta^- = [-1]^*\Theta$. Moreover, write $z = \sum_i (c_i) - \sum_i (d_i)$, with $c_i, d_i \in X$ and let $\mathfrak{a} = \iota_* z = \sum_i (\iota(c_i)) - \sum_i (\iota(d_i)) \in Z^0_0(J)$. We immediately find that

$$\langle \mathfrak{a}, \Theta^-_{b_1} - \Theta^-_{b_2}\rangle_v = \langle z, w\rangle_v \tag{34}$$

for all v, where the left hand side is defined in (27) (respectively (29)) and the right hand side is defined in (32) (respectively (33)) if $v \nmid p$ (respectively $v \mid p$).

Proposition 6.8. *For z and w as above, let $[z] = a \in J(K)$ and $[w] = b \in J(K)$. Then*

$$\hat{h}(a, \phi_\Theta(b)) = -\sum_v \langle z, w\rangle_v\,.$$

Proof. By construction, we have $a = S(\iota_* z) = S(\mathfrak{a})$, where $S\colon Z^0_0(J) \to J$ is the summation map, and $\phi_\Theta(b) = [\Theta_{b_1} - \Theta_{b_2}]$. Using Corollary 6.7, we obtain

$$\hat{h}(a, \phi_\Theta(b)) = -\langle \mathfrak{a}, \Theta^-_{b_1} - \Theta^-_{b_2}\rangle = -\sum_v \langle \mathfrak{a}, \Theta^-_{b_1} - \Theta^-_{b_2}\rangle_v = -\sum_v \langle z, w\rangle_v,$$

because $[-1]$ acts on $\mathrm{Pic}^0(A)$ as multiplication by -1. □

6.2. **Coleman-Gross.** As in §6.1.4, let X/K be a nice curve with Jacobian J and let $\iota\colon X \to J$ be an Abel-Jacobi map defined over K. Let Θ be the associated theta divisor. We now show that for appropriate choices, our height pairing on J is the same as the one constructed by Coleman and Gross [CG89]. In fact, to be slightly more general and to avoid the need to translate between Vologodsky and Coleman integration, we will compare our height pairing with the one constructed in [Bes17], which is the same as the Coleman-Gross height pairing with Coleman integration replaced by Vologodsky integration, so that it applies to curves with bad reduction above p as well. We will nevertheless continue to call this the Coleman-Gross height pairing.

The Coleman-Gross p-adic height pairing

$$h^{CG}\colon J(K) \times J(K) \to \mathbb{Q}_p$$

depends, in addition to χ, on the choice, for every $\mathfrak{p}|p$, of a splitting of the Hodge filtration on $H^1_{\mathrm{dR}}(X_{K_\mathfrak{p}})$; in other words, a subspace $W_\mathfrak{p}$ of $H^1_{\mathrm{dR}}(X_{K_\mathfrak{p}})$ complementary to the image of the holomorphic differentials.

The goal of this section is to prove the following comparison result.

Theorem 6.9. *For $\mathfrak{p} \mid p$, let $\alpha_{\mathfrak{p}}$ be a choice of curvature form of $\mathcal{P}_{\mathfrak{p}}$ and let $W_{\mathfrak{p}}$ be the complementary subspace associated to $\alpha_{\mathfrak{p}}$ in Proposition 6.11 below. Write $\underline{\alpha} = (\alpha_{\mathfrak{p}})_{\mathfrak{p}|p}$ and $\underline{W} = (W_{\mathfrak{p}})_{\mathfrak{p}|p}$. Relative to these choices, let $\hat{h} = \hat{h}_{\mathcal{P},\underline{\alpha},\chi}$ be the canonical height and let $h^{CG} = h^{CG}_{\underline{W},\chi}$ be the Coleman-Gross height pairing. Then we have*

$$\hat{h}(a, \phi_\Theta(b)) = -h^{CG}(a, b)$$

for all $a, b \in J(K)$.

The strategy of the proof is to decompose both pairings into local terms and to show that the local terms agree. The Coleman-Gross height is defined on a pair of divisors $z, w \in \mathrm{Div}^0(X)$ with disjoint support, as the finite sum

$$h^{CG}(z, w) = \sum_v h_v^{CG}(z, w) \; ,$$

over all finite primes v of K, of local height pairings. For every such v the local height pairing $h_v^{CG}(z, w)$ depends only on the images of z, w in $\mathrm{Div}^0(X_{K_v})$ To prove Theorem 6.9 it suffices, by Proposition 6.8, to show that $h_v^{CG}(z, w) = \langle z, w\rangle_v$, where the latter is defined in (32) (respectively (33)) if $v \nmid p$ (respectively $v \mid p$).

Remark 6.10. For $\mathfrak{q} \nmid p$, the local Coleman-Gross pairing between divisors $z, w \in \mathrm{Div}^0(X_{K_{\mathfrak{q}}})$ with disjoint support is the unique pairing satisfying the conditions of Lemma 6.1 (see [CG89, Section 2]), so it is equal to our pairing $\langle z, w\rangle_{\mathfrak{q}}$.

It remains to consider places $\mathfrak{p} \mid p$. The local Coleman-Gross pairing $\langle z, w\rangle_{\mathfrak{p}}$, for $z, w \in \mathrm{Div}^0(X_{K_{\mathfrak{p}}})$ with disjoint support is given in this case as follows: There is a map

$$\Psi : \Omega^1_{K_{\mathfrak{p}}(X_{\mathfrak{p}})} \to H^1_{\mathrm{dR}}(X_{\mathfrak{p}}) \; ,$$

where the source of the map is the space of meromorphic one-forms on $X_{\mathfrak{p}}$. This map, constructed in [Bes05, Definition 3.9] (there it is called Ψ') is as extension of the map Ψ considered in [CG89], which is the logarithm of the universal vectorial extension of the Jacobian of X. It has the following two properties:

$$\Psi(\eta) = [\eta] \text{ where } [\eta] \text{ is the cohomology class of the form of the second kind } \eta. \tag{35}$$

$$\Psi d\log(f) = 0 \text{ for any rational function } f \; . \tag{36}$$

It easily follows that corresponding to the divisor w there exists a unique meromorphic form ω_w with the properties

(1) the form ω_w has log singularities and its residue divisor is w;
(2) we have $\Psi(\omega_w) \in W_{\mathfrak{p}}$.

The local height pairing is given by

$$h_{\mathfrak{p}}^{CG}(z, w) = t_{\mathfrak{p}}(\int_z \omega_w) \; ,$$

where $t_{\mathfrak{p}}$ and the branch of the logarithm used for integration are obtained from (24).

For the same $z, w \in \mathrm{Div}^0(X_{K_{\mathfrak{p}}})$ as above, our local pairing $\langle z, w\rangle_{\mathfrak{p}}$ defined in (33) depends on the choice of a curvature form $\alpha_{\mathfrak{p}}$. In order to compare this with the local Coleman-Gross height pairing $h_{\mathfrak{p}}^{CG}(z, w)$ we first have to compare this choice with the choice of a splitting of the Hodge filtration on $H^1_{\mathrm{dR}}(X_{K_{\mathfrak{p}}})$ needed for the Coleman-Gross pairing. This is what we will do in the next subsection.

6.2.1. *Curvature forms and complementary subspaces.* For notational ease, let $F = K_{\mathfrak{p}}$. Let A be an abelian variety over F. Recall the notion of purely mixed curvature forms from Definition 4.22 and the discussion preceding it. It is well known (see for example [Col98, p. 1380]) that there is a natural duality between $H^1_{\mathrm{dR}}(A)$ and $H^1_{\mathrm{dR}}(\hat{A})$. By [BBM82, Lemma 5.1.4], $\Omega^1(\hat{A})$ is precisely the annihilator of $\Omega^1(A)$ under the pairing. In the next result, we use the notation introduced before Definition 4.22.

Proposition 6.11. *There is a one to one correspondence between purely mixed curvature forms α on $\mathcal{P}$ and complementary subspaces to $\Omega^1(A)$ in $H^1_{\mathrm{dR}}(A)$ which is given as follows:*

(1) For α as above, the element $\alpha_{21} \in H_{21} \cong \Omega^1(\hat{A}) \otimes H^1_{\mathrm{dR}}(A)$ gives a linear map from $\mathrm{Hom}(\Omega^1(\hat{A}), F)$ to $H^1_{\mathrm{dR}}(A)$ and the corresponding subspace W_α is its image.

(2) *Given a complementary subspace W, let W' be its annihilator in $H^1_{\mathrm{dR}}(\hat{A})$. Then W is dual to $\Omega^1(\hat{A})$ while W' is dual to $\Omega^1(A)$. The dualities provide elements*

$$\alpha_{21} \in \mathrm{Hom}(\mathrm{Hom}(\Omega^1(\hat{A}, F), W)) \cong \Omega^1(\hat{A}) \otimes W \subset \Omega^1(\hat{A}) \otimes H^1_{\mathrm{dR}}(A) \cong H_{21} ,$$

$$\alpha_{12} \in \mathrm{Hom}(\mathrm{Hom}(\Omega^1(A, F), W')) \cong \Omega^1(A) \otimes W' \subset \Omega^1(A) \otimes H^1_{\mathrm{dR}}(\hat{A}) \cong H_{12} ,$$

and the corresponding curvature form is $\alpha_W = \alpha_{12} + \alpha_{21}$.

Explicitly, the form α_W may be written as follows: Choose bases

$$\omega_k^1,\ \omega_k^2,\ k = 1, \ldots, g \tag{37}$$

for $\Omega^1(A)$ and $\Omega^1(\hat{A})$, respectively. Take a basis η_k^1 ($k = 1, \ldots, g$) of W which is dual to the basis ω_k^2 of $\Omega^1(\hat{A})$ and a basis η_k^2 of W' which is dual to the basis ω_k^1. In this way, the ω_k^1 and η_k^1 form a basis for $H^1_{\mathrm{dR}}(A)$ and the dual basis for $H^1_{\mathrm{dR}}(\hat{A})$ is provided by the η_k^2 and ω_k^2. The curvature form corresponding to W is then given by

$$\alpha_W = \sum_k \pi_1^* \omega_k^1 \otimes \pi_2^*[\eta_k^2] - \pi_2^* \omega_k^2 \otimes \pi_1^*[\eta_k^1] . \tag{38}$$

Proof. It is easy to see that (38) is just an explicit form of the description of α_W in terms of W. We recall that the relation between $\mathcal{P}$ and the duality is that $\mathrm{ch}_1(\mathcal{P})$ lies in $H^1_{\mathrm{dR}}(A) \otimes H^1_{\mathrm{dR}}(\hat{A})$, embedded in $H^2_{\mathrm{dR}}(A \times \hat{A})$ by the Kunneth formula and this class gives the required duality. In concrete terms, this means that with respect to any choice of dual bases for $H^1_{\mathrm{dR}}(A)$ and $H^1_{\mathrm{dR}}(\hat{A})$, in particular the bases of ω's and η's we chose before, we have

$$\mathrm{ch}_1(\mathcal{P}) = \sum_k (\pi_1^* \omega_k^1 \cup \pi_2^* \eta_k^2 + \pi_1^* \eta_k^1 \cup \pi_2^* \omega_k^2) .$$

This immediately shows that the curvature form α_W we have associated with W indeed cups to $\mathrm{ch}_1(\mathcal{P})$.

Now fix one complementary subspace W_0 and choose bases corresponding to W_0 as above, and write, in terms of this basis, a general curvature form satisfying our two conditions

$$\alpha = \sum_{i,j,k,l} a_{ij}^{kl} \pi_i^* \omega_k^i \otimes \pi_j^*[\omega_l^j] + \sum_{i,j,k,l} b_{ij}^{kl} \pi_i^* \omega_k^i \otimes \pi_j^*[\eta_l^j] \text{ such that } \alpha_{11} = \alpha_{22} = 0 \text{ and } \cup \alpha = \mathrm{ch}_1(\mathcal{P}) , \tag{39}$$

where the a_{ij}^{kl} and b_{ij}^{kl} are constants for which we now need to find restrictions. The fact that α cups to $\mathrm{ch}_1(\mathcal{P})$ implies that $b_{12}^{kk} = 1$ and $b_{21}^{kk} = -1$ for all k and all other b_{ij}^{kl} vanish and that $a_{ij}^{kl} = a_{ji}^{lk}$ for all possible indices. The condition $\alpha_{11} = \alpha_{22} = 0$ implies that $a_{11}^{kl} = a_{22}^{kl} = b_{11}^{kl} = b_{22}^{kl} = 0$ for all k, l. For an α as in (39), let us now compute the corresponding W_α. We can write

$$\alpha_{21} = \sum_k \omega_k^2 \otimes \overline{\omega}_k$$

for appropriate classes

$$\overline{\omega}_k = -[\eta_k^1] + \sum_l a_{21}^{kl} [\omega_l^1] \in H^1_{\mathrm{dR}}(A)$$

and $W_\alpha = \mathrm{Span}(\overline{\omega}_1, \ldots, \overline{\omega}_g)$. As the spanning vectors for W_α are congruent modulo $\Omega^1(A)$ to the basis elements $[\eta_k^1]$, it is clear that W_α is indeed complementary.

To complete the proof, it remains to show that the two constructions are inverses one one another. It is immediate that $W_{\alpha_W} = W$. To see that also $\alpha_{W_\alpha} = \alpha$ we now repeat the computation above of a general curvature form, but this time we start, instead of the basis provided by W_0, with the basis provided by $W = W_\alpha$ itself. With this choice of basis it is now immediate that all a_{ij}^{kl} are 0 and therefore $\alpha = \alpha_W$. □

6.2.2. *Forms of third kind with prescribed residues from log functions.* Continuing with the notation of the previous subsection, suppose now that the abelian variety is $J = \mathrm{Jac}(X)$, the Jacobian of a nice curve X over $F = K_{\mathfrak{p}}$. There is an isomorphism $H^1_{\mathrm{dR}}(X) \cong H^1_{\mathrm{dR}}(J)$, compatible with the Hodge filtration. Hence the choice of a complementary subspace $W \in H^1_{\mathrm{dR}}(X)$, as required for the construction of the local Coleman-Gross pairing, determines a complementary subspace in $H^1_{\mathrm{dR}}(J)$. We denote this subspace by W. By Proposition 6.11, this choice determines a curvature form α on the Poincaré line bundle on $J \times \hat{J}$. Let $w \in \mathrm{Div}^0(X)$ and suppose that L is a line bundle on X and s a rational section of L such that $\mathrm{div}(s) = w$. Then α induces a canonical log function $\log_L$ and a local height pairing $\langle \cdot, \cdot \rangle_{\mathfrak{p}}$ as in (33). The function $\log_L(s)$ is locally analytic outside the support of w and $d \log_L(s)$ is independent of the ambiguity in $\log_L$.

Proposition 6.12. *In the situation above we have $d \log_L(s) = \omega_w$.*

By the construction of the two local height pairings it is clear that Proposition 6.12 implies:

Corollary 6.13. *We have* $h^{CG}_{\mathfrak{p}}(z,w) = \langle z,w\rangle_{\mathfrak{p}}$.

Proof of Theorem 6.9. This follows from Remark 6.10, Corollary 6.13 and Proposition 6.8. □

Proof of Proposition 6.12. The metrized line bundle $(L, \log_L)$ is flat, because $\log_L$ is the pullback of a log function on a line bundle on J, which is flat by Theorem 4.18 and Corollary 4.27 because α is purely mixed. Thus, the one form $d\log_L(s)$ is meromorphic, and it has simple poles with residue divisor exactly w by Lemma 3.12. Proving that $d\log_L(s) = \omega_w$ is therefore equivalent to showing that $\Psi(d\log(s)) \in W$. For a given line bundle L of degree 0, the class $\Psi(d\log(s)) \in H^1_{\mathrm{dR}}(X)$ is independent of the choice of the section s, by (36), because for another s the form will differ by a $d\log$ of a rational function. The association

$$L \mapsto \Psi(d\log(s))$$

clearly maps tensor products to sums and therefore gives a homomorphism

$$r\colon J(F) \to H^1_{\mathrm{dR}}(X)\,,$$

which is locally analytic. The proposition follows if we can show that the image of r is contained in W.

We will follow closely the proof of Theorem 7.3 in [Bes05]. Since r is locally analytic it suffices to show that its derivative with respect to any vector field on J at one, hence at any point, is in W. The computation is essentially the same as in the proof of Theorem 7.3 in [Bes05]. Let $\mathcal{P}_X$ be the pullback of $\mathcal{P}$ to $X \times J$ via the map $X \times J \to J \times J$. Let α_X be the curvature of $\mathcal{P}_X$ induced by pullback of α. It has exactly the same formula as (38) under the identification of the cohomologies and forms of X and J. To compute the derivative of r with respect to a vector field T on J, we pick a rational section s defined on an open $U \in X \times J$. Locally, in the analytic topology on J, the relative form $d\log_{\mathcal{P}_X}(s)$ is a family of forms of the third kind on X in the sense of [Bes05, Definition A.1] (called there relative forms of the third kind). We need to apply Ψ fiber by fiber and then differentiate with respect to T. However, because the derivative of a family of forms of the third kind is a form of the second kind by Definition A.3 in [Bes05] and the following discussion, and these are sent by Ψ to their associated cohomology classes according to (35), we can instead differentiate with respect to T and then take the cohomology class fiber by fiber. We now repeat the computation from the proof of [Bes05, Theorem 7.3], adjusting notation. We need to compute $\partial_T d\log_{\mathcal{P}_X}(s)$. This is the same as $d(d\log_{\mathcal{P}_X}(s)|_{\partial_T})$, where the last notation means retraction in the direction of T. Then, just as in the above reference, we notice that $\overline{\partial}(d\log_{\mathcal{P}_X}(s)|_{\partial_T}) = \overline{\partial}(d\log_{\mathcal{P}_X}(s)|_{\partial_T})$ with the retraction acting on the first coordinate of the tensor product. The term $\overline{\partial}(d\log_{\mathcal{P}_X}(s)$ is precisely the curvature α_X restricted to U. Just as in [Bes05] after retraction and restriction to the fiber of the projection to J, the only term that survives is

$$\sum \omega_k^2|_{\partial_T}\overline{\omega_k}\,.$$

Again as in [Bes05], this shows that the required cohomology class lies in the subspace spanned by the $\overline{\omega}_k$, which is W. □

7. Quadratic Chabauty

Let $X/\mathbb{Q}$ be nice curve of genus $g > 1$ such that $X(\mathbb{Q}) \neq \varnothing$ and let p be a prime number. Our goal in this section is to use the theory of (canonical) p-adic heights developed above (in particular, the properties of the local terms) to construct a nonconstant locally analytic function on $X(\mathbb{Q}_p)$ whose values on $X(\mathbb{Q})$ can be controlled. Fixing a base point $b \in X(\mathbb{Q})$, we let $\iota = \iota_b\colon X \hookrightarrow J$ denote the corresponding Abel-Jacobi map, inducing a morphism $\mathrm{NS}(J) \to \mathrm{NS}(X)$, where J is the Jacobian of X. We fix a purely mixed curvature form α on $\mathcal{P}_p$. We assume that $\ker(\mathrm{NS}(J) \to \mathrm{NS}(X))$ has positive rank.

Remark 7.1. The assumption that $\ker(\mathrm{NS}(J) \to \mathrm{NS}(X))$ has positive rank, combined with the surjectivity of $\iota^*|_{\mathrm{Pic}^0(X)}$ guarantees that there is some line bundle L on J such that $\iota^*L = \mathcal{O}_X$. Henceforth we fix such a line bundle L.

Since $\iota^*L = \mathcal{O}_X$, we get a map on total spaces of line bundles $\tilde{\iota}\colon X \times \mathbb{G}_m \to L^\times$. We let 1 denote the section $x \mapsto (x,1)$ of the natural projection $X \times \mathbb{G}_m \to X$. We endow L with the rigidification $r_b := \tilde{\iota}(1(b))$; the corresponding canonical adelic metric $(\log_{L,p}, \{v_{L,q}\}_q)$ then satisfies

$$v_{L,q}(r_b) = 0 \tag{40}$$

for all $q \neq p$ and

$$\log_{L,p}(r_b) = 0\,. \tag{41}$$

We write $\hat{h}_L$ for the corresponding p-adic height. By pullback, we obtain an adelic metric on $\mathcal{O}_X$:

$$(\log_p, \{v_q\}_q) := \iota^*(\log_{L,p}, \{v_{L,q}\}_q)\,. \tag{42}$$

For a prime $q \neq p$, the valuation v_q is a $\mathbb{Q}$-valuation (see Remark 2.7) and we define a local height function

$$\lambda_q \colon X(\mathbb{Q}_q) \to \mathbb{Q}\,; \quad x \mapsto v_q(1(x))\,.$$

Lemma 7.2. *Let $q \neq p$ be prime. Then λ_q*

(1) takes only finitely many values;
(2) is identically 0 if X has potentially good reduction at q.

Proof. Since $v_{L,q}$ is a good valuation on L, Proposition 2.9 implies that λ_q is a locally constant $\mathbb{Q}$-valued function on $X(\mathbb{Q}_p)$, implying the first statement. For the second, we may assume that J has good reduction at q by passing to an extension. Then a good valuation on L is a model valuation on the Néron model $\mathcal{J}$ by Remark 2.12. The closure $\mathcal{L}$ of L on $\mathcal{J}$ pulls back to an extension of $\mathcal{O}_X$ to a semistable model $\mathcal{X}$ of X, so v_q is constant, hence identically 0 by (40). □

Henceforth we assume, in addition to $\operatorname{rk}\operatorname{NS}(J) > 0$, that $\operatorname{rk} J(\mathbb{Q}) = g$. We also assume that p is a prime number such that the p-adic closure of $J(\mathbb{Q})$ has finite index in $J(\mathbb{Q}_p)$. Fix a basis $(\omega_0, \dots, \omega_{g-1})$ of $H^0(J, \Omega^1)$. By abuse of notation, we also write ω_i for $\iota^*\omega_i$. Our assumptions on J imply that $(J(\mathbb{Q}) \otimes \mathbb{Q}_p)^\vee$ is generated by $f_0, \dots, f_{g-1}$, where $f_i(x) = \log(x)(\omega_i) = \int_0^x \omega_i$. By Proposition 5.5, the height $\hat{h}_L$ is a quadratic polynomial in the f_i with constant term 0, say

$$\hat{h}_L = \sum a_{ij} f_i f_j + \sum b_k f_k\,. \tag{43}$$

The idea of the Quadratic Chabauty method is to solve for the constants a_{ij} and b_k and to use that for $x \in X(\mathbb{Q})$, Equation (43) implies

$$\sum a_{ij} \int_b^x \omega_i \int_b^x \omega_j + \sum b_k \int_b^x \omega_k - \log_p \circ 1(x) = \sum_{q \neq p} \lambda_q(x) \chi_q(q)\,. \tag{44}$$

Theorem 7.3. *Suppose that*

$$\hat{h}_L = \sum a_{ij} f_i f_j + \sum b_k f_k$$

for constants $a_{ij}, b_k \in \mathbb{Q}_p$. Then

$$F \colon X(\mathbb{Q}_p) \to \mathbb{Q}_p\,; \quad x \mapsto \sum a_{ij} \int_b^x \omega_i \int_b^x \omega_j + \sum b_k \int_b^x \omega_k - \log_p \circ 1(x)$$

is a Vologodsky function. It takes values on $X(\mathbb{Q})$ in the finite set $T = \{\sum_{q \neq p} l_q \cdot \chi_q(q)\}$, where l_q runs through the values that the function λ_q takes on $X(\mathbb{Q}_q)$. Moreover, for every $t \in T$ there are only finitely many points $x \in X(\mathbb{Q}_p)$ such that $F(x) = t$.

Proof. Since $\log_p$ is a Vologodsky function on $\mathcal{O}_X^\times$ and since $1\colon X(\mathbb{Q}_p) \to \mathcal{O}_X^\times(\mathbb{Q}_p)$ has no poles, the function $\log_p \circ 1$ is indeed a Vologodsky function on all of $X(\mathbb{Q}_p)$. Hence the same is true for F.

If $x \in X(\mathbb{Q})$, then we have

$$F(x) = \hat{h}_L(\iota(x)) - \log_p \circ 1(x) = \sum_{q \neq p} \lambda_q(x)\chi_q(q)\,.$$

The finiteness of T follows from Lemma 7.2.

To prove the final claim, note that a Vologodsky function is locally analytic. If it obtains a value an infinite number of times on a residue disc, then it must be constant on that disc, hence by the identity principle for Vologodsky functions [Bes05, Lemma 2.5] it must be constant. It is therefore sufficient to prove that F is not constant. The exact sequences (5) and (7) imply that it is sufficient to prove that

$$\overline{\partial} d(F) \neq 0. \tag{45}$$

First note that $\overline{\partial} d \log_p \circ 1 = \operatorname{Curve}(\log_p)$ as in the proof of [Bes05, Proposition 4.4]. But $F - \log_p \circ 1$ is a product of integrals of holomorphic forms and thus its $\overline{\partial} d$ resides in $\Omega^1(X) \otimes \Omega^1(X) \subset \Omega^1(X) \otimes H^1_{\mathrm{dR}}(X)$. It suffices then to prove that $\operatorname{Curve}(\log_p)$ is not in this subspace. As $\operatorname{Curve}(\log_p) = \iota^* \operatorname{Curve}(\log_L)$, it suffices to show that

$$\operatorname{Curve}(\log_L) \notin \Omega^1(J) \otimes \Omega^1(J)\,. \tag{46}$$

To show this, we use the Hodge filtration over $\mathbb{C}$. It is well-known that over $\mathbb{C}$, the Chern class of L is anti-invariant with respect to complex conjugation. Hence, it is not in $F^2 H^2_{\mathrm{dR}}(J)$, because $F^2 \cap \overline{F^2} = 0$. This proves (46), and hence completes the proof of the theorem. □

7.1. **Computing rational points using Quadratic Chabauty.** Let $X/\mathbb{Q}$ be a nice curve satisfying the conditions of Theorem 7.3. In order to compute the rational points on X, we need to

(I) compute $\lambda_p(x)$ for $x \in X(\mathbb{Q}_p)$ (see §7.1.1);
(II) find all possible values of $\lambda_q(x)$ for bad primes $q \neq p$ and $x \in X(\mathbb{Q}_q)$ (see §7.1.2);
(III) solve for the coefficients a_{ij} and b_k (see §7.1.3)
(IV) compute the set $\mathcal{Z} := \{z \in X(\mathbb{Q}_p) \,:\, F(z) \in T\}$;
(V) identify $X(\mathbb{Q})$ inside $\mathcal{Z}$.

These tasks are the same as in [BD18, BDM+19]. The final two steps are standard: Step (IV) is possible in practice using the Weierstrass preparation theorem. If $\mathrm{rkNS}(J) > 2$, we can repeat Steps (I)–(IV) for another line bundle L' on J such that $\iota^* L' \cong \mathcal{O}_X$ and such that the classes of L and L' in $\mathrm{NS}(J)$ are independent. We expect that the sets Z for L and L' will usually only have $X(\mathbb{Q})$ as common roots, unless there is a geometric reason for further common roots. Alternatively, we can combine $\mathcal{Z}$ with information at primes $q \neq p$ via the Mordell-Weil sieve [BS10, BBM17].

We briefly discuss Steps (I), (II) and (III) below, referring to future work for more details.

7.1.1. *Computing* λ_p. For Step (I), we need to expand the function $\lambda_p = \log_p \circ 1$ into a convergent power series on every residue disk of $X(\mathbb{Q}_p)$. Recall that $\log_p = \iota^* \log_{L,p}$ has curvature $\iota^* \alpha_L$, where

$$\alpha_L = \frac{1}{2}\alpha_{L^+} = \frac{1}{2}(\mathrm{id} \times \phi_L)^* \alpha\,.$$

However, α_L does not determine $\log_{L,p}$; in fact, it is not straightforward to construct *any* good log function on L_p. Fortunately, this is not necessary. Let $\log'_p$ be *some* log function on $\mathcal{O}_X$ with curvature $\iota^*\alpha_L$ and set $\lambda'_p := \log'_p \circ 1$. By Proposition 3.4, we have $\log'_p = \log_p + \int \theta$ for some unknown holomorphic form $\theta \in \Omega^1(X_p)$ which induces via ι^* a linear form

$$\ell\colon J(\mathbb{Q}_p) \to \mathbb{Q}_p \quad a \mapsto \int_0^a \theta\,.$$

We set

$$h' := \hat{h}_L + \ell\,.$$

Then Theorem 7.3 remains true if we replace $\hat{h}_L$ by h' and λ_p by λ'_p, without changing λ_q for $q \neq p$. Hence we can work with $\log'_p$ in place of $\log_p$ and there is no need to compute θ. An explicit construction of a log function $\log'_p$ on $\mathcal{O}_X$ with curvature form $i^*\alpha_L$ as an iterated integral is given in § 3.3.3. It thus remains to compute the curvature form α_L; we will describe in future work how this can be done in terms of a complementary subspace W and how this can be used to compute $\log'_p$ in practice.

7.1.2. *Computing (all possible values of)*λ_q. Let $q \neq p$ be a prime of bad reduction for X. We would like to compute a finite set $T_q \subset \mathbb{Q}$ such that $\lambda_q(X(\mathbb{Q}_q)) \subset T_q$. We first strengthen Lemma 7.2.

Proposition 7.4. *Let $K/\mathbb{Q}_q$ be a finite extension such that X has a semistable regular model $\mathcal{X}$ over $\mathcal{O}_K$. Then $\lambda_q(P)$ only depends on the component of the special fiber $\mathcal{X}_q$ that P reduces to.*

Proof. Let $\mathcal{C}$ be a component of the special fiber $\mathcal{J}_q$ of the Néron model of J over $\mathcal{O}_K$. It suffices to show that if x and y are points in $X(K)$ such that $\iota(x)$ and $\iota(y)$ both reduce to $\mathcal{C}$, then we have $\lambda_q(x) = \lambda_q(y)$.

Let s be a meromorphic section of L such that $\iota(x)$ and $\iota(y)$ are not in the support of $D := \mathrm{div}(s)$. By [Lan83, Theorem 11.5.1], there is a constant γ_s such that for all $a \in J(K) \setminus \mathrm{supp}(D)$ we have

$$v_{L,q}(s(a)) = i(D,a) + \gamma_s\,, \tag{47}$$

where $i(D,a)$ is the intersection multiplicity of the extensions of D and a on $\mathcal{J}$ defined before [Lan83, Theorem 11.5.1]. Since $\iota^*L = \mathcal{O}_X$, the section s pulls back to a rational function f on X. We then have

$$\begin{aligned}\lambda_q(x) - \lambda_q(y) &= v_{L,q}(s(\iota(x)) - v_{L,q}(s(\iota(y)) - (\mathrm{ord}_q(f(x)) - \mathrm{ord}_q(f(y)))\\ &= i(D,\iota(x)) - i(D,\iota(y)) - (\mathrm{ord}_q(f(x)) - \mathrm{ord}_q(f(y)))\,.\end{aligned}$$

To show that this vanishes, we use the fact that ι induces a morphism

$$\bar{\iota}\colon \mathcal{X}^{sm} \to \mathcal{J}\,,$$

where $\mathcal{X}^{sm}$ is the smooth locus of $\mathcal{X}$. Denoting the closure of D on $\mathcal{J}$ by $\mathcal{D}$ and the closure of $D' := \mathrm{div}(f)$ on $\mathcal{X}$ by $\mathcal{D}'$, we then have $\bar{\iota}^*\mathcal{D} = \mathcal{D}' + V'_s$ for some vertical divisor V'_s on $\mathcal{X}'$. We extend V'_s to a vertical divisor V_s on $\mathcal{X}$. If $P \in X(K)$ with closure $\mathcal{P}$ on $\mathcal{X}$, then the projection formula implies

$$i(D,\iota(P)) = (\mathcal{D}\,.\,\bar{\iota}(P)) = (\mathcal{D}'\,.\,\mathcal{P}) + (V_s\,.\,\mathcal{P}) = \mathrm{ord}_q(f(P)) + (V_s\,.\,\mathcal{P})\,.$$

Since the intersection multiplicity $(V_s\,.\,\mathcal{P})$ only depends on the component of $\mathcal{X}_q$ that P reduces to, we deduce that $\lambda_q(x) = \lambda_q(y)$. □

By the proof of Proposition 7.4, it suffices to find, for every component $\mathcal{C}$ of $\mathcal{J}_q$, one point $x \in J(K)$ reducing to $\mathcal{C}$ and to compute $v_{L,q}(s(z))$ for some section s of L such that $x \notin \operatorname{div}(s)$. However, this is a difficult problem in general. Even for $L = \Theta$, it is only known how to compute the canonical valuation explicitly for hyperelliptic curves of genus ≤ 3 (see [FS97, Sto02, MS16, Sto17]).

Remark 7.5. In future work, we will describe the pullback of the canonical valuation $v_{L,q}$ in terms of iterated Vologodsky integrals directly on the curve. We will show how this may be used to compute λ_q in practice.

7.1.3. *Solving for the height.* We briefly discuss possible approaches to the computation of the constants a_{ij} and b_k in Theorem 7.3. The line bundle L induces an endomorphism $E_L = \phi_\Theta^{-1} \circ \phi_L$ on J, and hence on $H^1_{\mathrm{dR}}(X) \cong H^1_{\mathrm{dR}}(J)$. We set $J_p := J(\mathbb{Q}) \otimes \mathbb{Q}_p$. If there are enough points in $X(\mathbb{Q})$ such that their images generate $(J_p \otimes J_p) \oplus J_p$ under the embedding $(\iota \otimes E_L \circ \iota + [L^-], \iota)$, then we can compute the constants a_{ij} and b_k by computing $\hat{h}_L$ (or h') and the functions f_i and $f_i f_j$ in these images.

Alternatively, note that the bilinear pairing associated to $\hat{h}_L$ and h' is the same, say

$$B(x,y) := \frac{1}{2}(h'(x+y) - h'(x) - h'(y)) = \frac{1}{2}(h_L(x+y) - h_L(x) - h_L(y))\,.$$

We want to write B in terms of the basis of $(J(\mathbb{Q}) \otimes J(\mathbb{Q}) \otimes \mathbb{Q}_p)^*$ given by

$$(g_{ij})_{0 \leq i \leq j < g}\,, \quad g_{ij} = \frac{1}{2}(f_i f_j + f_j f_i))$$

by evaluating B and the g_{ij} in sufficiently many points. We conclude that

$$B(x,y) = \hat{h}_\Theta(x, E_L(y))\,. \tag{48}$$

If we evaluate $\hat{h}_\Theta$ in sufficiently many points, we get an expression

$$\hat{h}_\Theta(x,y) = \sum_{i,j} c_{ij}(f_i f_j + f_j f_i)(x,y)$$

and we obtain the constants a_{ij} by evaluating, using $f_i(E_L(y)) = \int_0^{E_L(y)} = \int_0^y E_L^*(\omega_i)$. Since $B(x,x) = \hat{h}_{L^+}(x)$ is the quadratic term of both $\hat{h}_L$ and h', the linear part of h' is $h'(x) - B(x,x)$, and we can compute it by evaluating in sufficiently many points. This gives us the constants b_k.

8. Comparison with Balakrishnan and Dogra's approach to Quadratic Chabauty

In this section we clarify the relation between our Quadratic Chabauty construction and the original one of Balakrishnan and Dogra [BD18]. Using the equivalence between the height pairings of Nekovář and of Coleman-Gross, their local contributions are of the form

$$h_v^{CG}(D(b,z), z - b),$$

where $D(b,z)$ is a divisor on X constructed using the line bundle L equipped with a section. It turns out that we use the same global height (see Proposition 8.5).

We first spell out the construction of the divisor $D(b,z) \in \operatorname{Div}(X)$ starting from a divisor in the class of the line bundle $L \in \operatorname{Pic}(J)$. For this, we modify the construction of the divisor $D_Z(b)$ in [DF21, Section 2.2]. The divisor $D_Z(b)$ of [DF21] corresponds to the diagonal cycle $D(b,b)$ in our notation.

8.1. **Construction of the divisor** $D(b,z)$**.** Let X be a nice curve of genus $g > 1$ over a field K such that $X(K) \neq \varnothing$. Fix a base point $b \in X(K)$. Let $z \in X(\overline{K})$. We then have natural maps $i_{1,b}, i_{2,z}, \Delta\colon X \to X \times X$, defined as follows. We denote by Δ the diagonal embedding. Let $i_{1,b}\colon X \to X \times X$ be the map defined by $i_{1,b}(x) := (x,b)$. Similarly, we define $i_{2,z}\colon X \to X \times X$ by $i_{2,z}(x) = (z,x)$.

Let $\iota = \iota_b\colon X \to J$ be the Abel-Jacobi map with respect to the base point b and let $m\colon J \times J \to J$ denote the group law on the Jacobian. We define $\iota^{(2)}\colon X \times X \to J$ to be the composition $m \circ (\iota, \iota)$. Let $\phi_z\colon \operatorname{Pic}(X \times X) \to \operatorname{Pic}(X)$ be the map $\phi_z := \Delta^* - i_{1,b}^* - i_{2,z}^*$, where $\Delta^*\colon \operatorname{Pic}(X \times X) \to \operatorname{Pic}(X)$ is the pullback map on line bundles induced by $\Delta\colon X \to X \times X$, etc.

Definition 8.1. Let $L \in \mathrm{Pic}(J)$ and let s be a section of L. Let $Z := (\iota^{(2)})^*(L, s) \in \mathrm{Div}(X \times X)$. Define $D(b, z)$ to be the divisor corresponding to the image of (L, s) under the composition

$$\mathrm{Pic}(J) \xrightarrow{(\iota^{(2)})^*} \mathrm{Pic}(X \times X) \xrightarrow{\phi_z} \mathrm{Pic}(X).$$

We will denote the composite map $\theta_{X,b,z}$.

Balakrishnan and Dogra have an intersection-theoretic condition on the divisor Z as above [BD18, Definition 6.2, § 6.3]. They use this condition to justify that certain mixed extensions of Galois representations constructed out of D are isomorphic (see the end of [BD18, § 6.3] for more details). We now prove that their condition is equivalent to the condition that $\deg(\iota^* L) = 0$, our key condition for running our approach to Quadratic Chabauty.

Lemma 8.2. *The condition* $Z.(\Delta - X \times P_1 - P_2 \times X) = 0$ *for all* $P_1, P_2 \in X$ *is equivalent to the condition that* $\deg(\iota^* L) = 0$.

Proof. We may assume that $P_1 = P_2 = b$ without any loss of generality. Using the projection formula in the first line, $Z = (\iota^{(2)})^*(L, s)$ in the second line, the identities

$$\iota^{(2)} \circ i_\Delta = [2] \circ \iota, \qquad \iota^{(2)} \circ i_{1,b} = \iota^{(2)} \circ i_{2,b} = \iota$$

in the third line, and the isomorphisms from Equations (22), (15) and (16)

$$[2]^*(L^{\otimes 2}) = [2]^*(L^+) \otimes [2]^*(L^-) = (L^+)^{\otimes 4} \otimes L^{-\otimes 2} = (L^+)^{\otimes 2} \otimes L^{\otimes 2}$$

in the fourth line, we get

$$\begin{aligned} Z.(\Delta - X \times \{b\} - \{b\} \times X) &= \deg(\Delta^*(Z)) - \deg(i_{1,b}^*(Z)) - \deg(i_{2,b}^*(Z)) \\ &= \deg((\Delta^* \iota^{(2)^*} L) \otimes (i_{1,b}^* \iota^{(2)^*} L)^{-1} \otimes (i_{2,b}^* \iota^{(2)^*} L)^{-1}) \\ &= \deg(\iota^*([2]^* L \otimes L^{\otimes -2})) \\ &= \frac{\deg(\iota^* L^+)}{2}. \end{aligned}$$

Note that we are allowed to divide by 2, because $\mathrm{NS}(X) \cong \mathbb{Z}$ is torsion-free. Since $\deg(\iota^* L) = \deg(\iota^*([-1]^* L))$, it follows that $\deg(\iota^* L^+) = 2 \deg(\iota^* L)$ and we are done. □

8.2. **Endomorphisms of J and $D(b, z)$.** We keep the notation of the previous subsection and we assume, in addition, that $\iota^* L \cong \mathcal{O}_X$. Next, we modify the arguments in [DF21, Section 2.1] to reinterpret the divisor class of $D(b, z)$ in Definition 8.1 in terms of the action of the endomorphism of J corresponding to the line bundle L acting on the divisor class $\iota(z)$ of $z - b$ (Lemma 8.4 (g)). Along the way, we show that we can identify the Chow-Heegner point/Diagonal cycle $D_Z(b)$ of [DF21, Section 2.2] with the pullback of L^- by ι (Lemma 8.4 (e)). We will then use Lemma 8.4 to prove Proposition 8.5, which is a comparison between the global Coleman-Gross height pairing between the divisors $z - b$ and $D(b, z)$, and our canonical height for the line bundle L at $\iota(z)$.

We first need some more notation. Let π_1, π_2 denote the standard projections $X \times X \to X$ to the first and second factors respectively. Let

$$\psi\colon \mathrm{Pic}(X \times X) \to \mathrm{End}(J)$$

denote the usual action on J of a correspondence on $X \times X$, where we first pull back a degree 0 divisor on X by π_1, then intersect it with the given class in $\mathrm{Pic}(X \times X)$ and then push forward this intersection by π_2. The map ψ is surjective, with kernel the fibral divisors corresponding to π_1, π_2.

We now describe a natural splitting of the exact sequence

$$0 \to \pi_1^*(\mathrm{Pic}(X)) \oplus \pi_2^*(\mathrm{Pic}(X)) \to \mathrm{Pic}(X \times X) \xrightarrow{\psi} \mathrm{End}(J) \to 0 \tag{49}$$

using the maps $i_{1,b}, i_{2,z}$. This splitting will then be used to show that the map $\theta_{X,b,z}$ from Definition 8.1 factors through $\mathrm{End}(J)$. The splitting is straightforward – since $\pi_2 \cdot i_{1,b}$ and $\pi_1 \cdot i_{2,z}$ are constant maps, we see that $\ker(i_{1,b}^*) \oplus \ker(i_{2,z}^*)$ is a natural complement to the image of $\pi_1^*(\mathrm{Pic}(X)) \oplus \pi_2^*(\mathrm{Pic}(X))$ in $\mathrm{Pic}(X \times X)$. This gives a natural isomorphism, which by similar abuse of notation as in [DF21], we call ψ_z^{-1} –

$$\psi_z^{-1}\colon \mathrm{End}(J) \xrightarrow{\cong} \ker(i_{1,b}^*) \oplus \ker(i_{2,z}^*).$$

Remark 8.3. Using the decompositions

$$\mathrm{Pic}^0(X\times X)=\pi_1^*(\mathrm{Pic}^0(X))\oplus\pi_2^*(\mathrm{Pic}^0(X)),$$

and

$$\mathrm{Pic}(X\times X)=\pi_1^*(\mathrm{Pic}(X))\oplus\pi_2^*(\mathrm{Pic}(X))\oplus\ker(i_{1,b}^*)\oplus\ker(i_{2,z}^*),$$

which in turn induce the decomposition

$$\mathrm{NS}(X\times X)=\pi_1^*(\mathrm{NS}(X))\oplus\pi_2^*(\mathrm{NS}(X))\oplus\ker(i_{1,b}^*)\oplus\ker(i_{2,z}^*),$$

we may view $\ker(i_{1,b}^*)\oplus\ker(i_{2,z}^*)$ as a subspace of $\mathrm{NS}(X\times X)$, if we wish to do so.

For any $a\in J$, let $m_a\colon J\to J$ denote the translation by a map. Let $[-1]$ denote the inversion map on J. For any $L\in\mathrm{Pic}(J)$, let $L^+:=L\otimes[-1]^*L$ and let $L^-:=L\otimes([-1]^*L)^{-1}$. Given $L\in\mathrm{Pic}(J)$, let $\phi_L\colon J\to\hat{J}$ be the map $a\mapsto m_a^*(L)\otimes L^{-1}$. If Θ is the theta divisor on J with respect to ι, then ϕ_Θ denotes the corresponding principal polarization.

Lemma 8.4. *Let* $\tilde{\theta}_{X,b,z}\colon\mathrm{End}(J)\to\mathrm{Pic}(X)$ *be the map* $\tilde{\theta}_{X,b,z}:=\phi_z\cdot\psi_z^{-1}$. *Let* $\tilde{\phi}\colon\mathrm{Pic}(J)\to\mathrm{End}(J)$ *be the map* $\tilde{\phi}(L):=\phi_\Theta^{-1}\cdot\phi_L$. *Then*

(a) $\psi\cdot(\iota^{(2)})^*=-\tilde{\phi}$.
(b) $-\theta_{X,b,z}=\tilde{\theta}_{X,b,z}\cdot\tilde{\phi}$.
(c) $\tilde{\theta}_{X,b,z}(\tilde{\phi}(L))=-[D(b,z)]$.
(d) $2\tilde{\phi}(L)=\tilde{\phi}(L^+)$.
(e) $[D(b,b)]=\phi_\Theta^{-1}(L^-)$.
(f) $[D(b,z)]-[D(b,b)]=-\iota^*(\phi_L([z-b]))=\tilde{\phi}(L)([z-b])$.
(g) $[D(b,z)]=\phi_\Theta^{-1}(L^-)+\tilde{\phi}(L)([z-b])$

Proof.

(a) The inverse of ϕ_Θ is $-\iota^*$, and after unwinding definitions (see [DF21, Section 2.1] for more details), this in turn implies that
$$\psi\cdot(\iota^{(2)})^*=-\tilde{\phi}.$$
(b) Let $L\in\mathrm{Pic}(J)$ and let $[Z]:=(\iota^{(2)})^*(L)\in\mathrm{Pic}(X\times X)$. Then, since ψ_z splits the exact sequence (49), there is a fibral divisor $[F]\in\pi_1^*(\mathrm{Pic}(X))\oplus\pi_2^*(\mathrm{Pic}(X))$ such that $\psi_z^{-1}\cdot\psi([Z])=[Z]+[F]$. Since ϕ_z is trivial on fibral divisors, and in particular on $[F]$, combining this with (a), it follows that
$$-\theta_{X,b,z}(L)=-\phi_z\cdot(\iota^{(2)})^*(L)=-\phi_z\cdot\psi_z^{-1}\cdot\psi\cdot(\iota^{(2)})^*(L)=\phi_z\cdot\psi_z^{-1}\cdot\tilde{\phi}(L)=\tilde{\theta}_{X,b,z}\cdot\tilde{\phi}(L).$$
(c) Part (c) follows from Part (b) and Definition 8.1.
(d) Note that L^- is antisymmetric, and antisymmetric line bundles form precisely the kernel of the natural map $\mathrm{Pic}(J)\to\mathrm{NS}(J)$. Since the map $\tilde{\phi}\colon\mathrm{Pic}(J)\to\mathrm{End}(J)$ factors as
$$\tilde{\phi}\colon\mathrm{Pic}(J)\to\mathrm{NS}(J)\to\mathrm{End}(J),$$
combining this with the previous sentence, it follows that $\tilde{\phi}(L^-)=0$. Applying the homomorphism $\tilde{\phi}$ to the identity $L^{\otimes 2}=L^+\otimes L^-$ and using the previous line, we get
$$2\tilde{\phi}(L)=\tilde{\phi}(L^{\otimes 2})=\tilde{\phi}(L^+\otimes L^-)=\tilde{\phi}(L^+)+\tilde{\phi}(L^-)=\tilde{\phi}(L^+).$$
(e) Let $[2]\colon J\to J$ denote the multiplication by 2 map on J. Since $\iota^{(2)}\cdot i_{1,b}=\iota^{(2)}\cdot i_{2,b}=\iota$ and $\iota^{(2)}\cdot\Delta=[2]\cdot\iota$, a direct computation with Definition 8.1 and the identity
$$[2]^*L=L^{\otimes 3}\otimes[-1]^*L$$
shows that
$$[D(b,b)]=\iota^*([2]^*L\otimes L^{\otimes -2})=\iota^*(L^+). \tag{50}$$
Since $\iota^*(L)=0$ by choice of $L\in\mathrm{Pic}(J)$, it follows that
$$0=(\iota^*L)^{\otimes 2}=\iota^*(L^{\otimes 2})=\iota^*(L^+)+\iota^*(L^-),$$
and therefore
$$\iota^*(L^+)=-\iota^*(L^-). \tag{51}$$

Combining Equations (50) and (51) with the equality $-\iota^* = \phi_\Theta^{-1}$, we get

$$[D(b,b)] = -\iota^*(L^-) = \phi_\Theta^{-1}(L^-).$$

(f) Combining Definition 8.1 with the identities $\iota^{(2)} \cdot i_{2,b} = \iota$ and $\iota^{(2)} \cdot i_{2,z} = m_{\iota(z)} \cdot \iota$ and once again using the equality $-\iota^* = \phi_\Theta^{-1}$, we get

$$[D(b,z)] - [D(b,b)] = (i_{2,b}^* - i_{2,z}^*)((\iota^{(2)})^* L) = -\iota^*(m_{\iota(z)}^* L \otimes L^{-1}) = -\iota^*(\phi_L(\iota(z))) = \tilde{\phi}(L)(\iota(z)).$$

(g) Add the equations in Parts (e) and (f). □

8.3. **Comparison of global heights.** Now let $K = \mathbb{Q}$. We now compare our approach to Quadratic Chabauty with the one used in [BD18]. We keep the notation of the previous subsection.

Proposition 8.5. *Let $\hat{h}_L$ be the canonical p-adic height with respect to a choice α of curvature on $\mathcal{P}$ and a continuous idelé class character $\chi\colon \mathbb{A}_K/K^* \to \mathbb{Q}_p$. Let $h^{CG}\colon \operatorname{Div}^0(X) \times \operatorname{Div}^0(X) \to \mathbb{Q}_p$ denote the global Coleman-Gross height pairing between degree 0 divisors relative to χ and the complementary subspace W corresponding to α as in Proposition 6.11. Then we have for all $z \in X(\mathbb{Q})$*

$$2\hat{h}_L(\iota(z)) = -h^{CG}(z-b, D(b,z))$$

Proof. We will use the notation of Lemma 8.4. Let $E_L := \tilde{\phi}(L) = \phi_\Theta^{-1} \cdot \phi_L \in \operatorname{End}(J)$. Let $\hat{a}$ be the point in $\operatorname{Pic}^0(J)$ corresponding to the antisymmetric line bundle L^-. By Lemma 8.4 (g), we have

$$[D(b,z)] = \phi_\Theta^{-1}(\hat{a}) + E_L(\iota(z)).$$

Hence Theorem 6.9 implies that

$$h^{CG}(z-b, D(b,z)) = -\hat{h}_{\mathcal{P}}(\iota(z), \hat{a} + \phi_L(\iota(z))).$$

Now, by Corollary 5.9 (a), it follows that

$$\hat{h}_{\mathcal{P}}(\iota(z), \hat{a} + \phi_L(\iota(z))) = 2\hat{h}_L(\iota(z)).$$

□

Recall that the approach of Balakrishnan-Dogra is based on the height $h^{CG}(z-b, D(b,z))$, whereas we use the height $\hat{h}_L(\iota(z))$ to set up Quadratic Chabauty. Proposition 8.5 shows that the approaches are closely related. In fact one can refine Proposition 8.5 to compare the local contributions to Quadratic Chabauty; we will do so in future work.

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