CONDUCTOR-DISCRIMINANT INEQUALITY FOR HYPERELLIPTIC CURVES IN ODD RESIDUE CHARACTERISTIC

ANDREW OBUS AND PADMAVATHI SRINIVASAN

Abstract. We prove a conductor-discriminant inequality for all hyperelliptic curves defined over discretely valued fields K with perfect residue field of characteristic not 2. Specifically, if such a curve is given by $y^2 = f(x)$ with $f(x) \in \mathcal{O}_K[x]$, and if $\mathcal{X}$ is its minimal regular model over $\mathcal{O}_K$, then the negative of the Artin conductor of $\mathcal{X}$ (and thus also the number of irreducible components of the special fiber of $\mathcal{X}$) is bounded above by the valuation of $\operatorname{disc}(f)$. There are no restrictions on genus of the curve or on the ramification of the splitting field of f. This generalizes earlier work of Ogg, Saito, Liu, and the second author.

The proof relies on using so-called *Mac Lane valuations* to resolve singularities of arithmetic surfaces, a technique that was recently introduced by Wewers and the first author.

Contents

1. Introduction 1
Notation and conventions 8
Acknowledgements 9
2. The discriminant bonus and regular models 9
3. A key reduction step 12
4. Mac Lane valuations 16
5. Mac Lane valuations, normal models and regular resolutions 21
6. Regular horizontal divisors 33
7. Discriminant bonus calculations 42
8. Comparing number of components and the discriminant bonus 56
9. Proof of the conductor-discriminant inequality in the irreducible case 57
10. Proof of the conductor-discriminant inequality in the reducible case 59
Appendix A. Examples 72
References 74

1. Introduction

In this paper, we prove a conductor-discriminant inequality for all hyperelliptic curves over discretely valued fields with perfect residue field of characteristic not 2.

Let K be a discretely valued field with perfect residue field k of characteristic not 2. Let $\mathcal{O}_K$ be the ring of integers of K. Let $\nu_K \colon K \to \mathbb{Z} \cup \{\infty\}$ be the corresponding discrete

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valuation. Let X be a smooth, projective, geometrically integral curve of genus $g \geq 1$ defined over K. Let $S = \operatorname{Spec} \mathcal{O}_K$. Let $\mathcal{X}$ be a proper, flat, regular S-scheme with generic fiber X. The Artin conductor associated to the model $\mathcal{X}$ is defined by

$$\operatorname{Art}(\mathcal{X}/S) = \chi(\mathcal{X}_{\overline{K}}) - \chi(\mathcal{X}_{\overline{k}}) - \delta,$$

where χ is the Euler characteristic for the étale topology and δ is the Swan conductor associated to the ℓ-adic representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$). The Artin conductor is a measure of degeneracy of the model $\mathcal{X}$; it is a non-positive integer that is zero precisely when $\mathcal{X}/S$ is smooth or when $g = 1$ and $(\mathcal{X}_k)_{\text{red}}$ is smooth. If $\mathcal{X}/S$ is a regular, semistable model, then $-\operatorname{Art}(\mathcal{X}/S)$ equals the number of singular points of the special fiber $\mathcal{X}_k$. Let $\operatorname{Art}(X/K)$ denote the Artin conductor associated to the minimal proper regular model of X over $\mathcal{O}_K$.

For hyperelliptic curves, there is another measure of degeneracy defined in terms of minimal Weierstrass equations. Assume that X is hyperelliptic. An integral Weierstrass equation for X is an equation of the form $y^2 = f(x)$ with $f(x) \in \mathcal{O}_K[x]$, such that X is birational to the plane curve given by this equation. The discriminant of such an equation is defined to be the non-negative integer $\nu_K(\operatorname{disc}'(f))$, where $\operatorname{disc}'(f)$ is the discriminant of f, thought of as a polynomial of degree $2\lceil \deg(f)/2 \rceil$ (note that this is the usual discriminant $\operatorname{disc}(f)$ whenever f is monic or $\deg(f)$ is even). A minimal Weierstrass equation is an equation for which the integer $\nu_K(\operatorname{disc}'(f))$ is as small as possible amongst all integral equations. We define the *minimal discriminant* $\Delta_{X/K}$ of X to be $\nu_K(\operatorname{disc}'(f))$ for the minimal Weierstrass equation. The minimal discriminant of X is zero precisely when the minimal proper regular model of X is smooth over S.

When $g = 1$, we have $-\operatorname{Art}(X/K) = \Delta_{X/K}$ by the Ogg-Saito formula [Sai88, p.156, Corollary 2]. When $g = 2$, Liu [Liu94, p.52, Théorème 1 and p.53, Théorème 2] shows that $-\operatorname{Art}(X/K) \leq \Delta_{X/K}$; he also shows that equality can fail to hold. In the second author's thesis [Sri15], Liu's inequality was extended to hyperelliptic curves of arbitrary genus assuming that the roots of f are defined over an *unramified* extension of K. In subsequent work [Sri19], the second author proved the same inequality assuming only that roots of f are defined over a *tame* extension of K. The main result of this paper is to extend this inequality to *all* cases away from residue characteristic 2.

Theorem 1.1. *Let X be a hyperelliptic curve of genus $g \geq 1$ over a discretely valued field K with perfect residue field of characteristic not equal to 2. Let $\Delta_{X/K}$ be the minimal discriminant of X and let $\operatorname{Art}(X/K)$ denote the Artin conductor of the minimal regular model of X. Then $-\operatorname{Art}(X/K) \leq \Delta_{X/K}$.*

In genus 1, the proof of the Ogg-Saito formula used the explicit classification of special fibers of minimal regular models of genus 1 curves. In genus 2, [Liu94] defines another discriminant that is specific to genus 2 curves, and compares both the Artin conductor and the minimal discriminant (our $\Delta_{X/K}$, which Liu calls Δ_0) to this third discriminant (which Liu calls $\Delta_{\min}$). This third discriminant $\Delta_{\min}$ is sandwiched between the Artin conductor and the minimal discriminant and is defined using a possibly non-integral Weierstrass equation such that the associated differentials generate the $\mathcal{O}_K$-lattice of global sections of the relative dualizing sheaf of the minimal regular model. It does not directly generalize to higher genus hyperelliptic curves (but see [Liu94, Definition 1, Remarque 9] for a related conductor-discriminant question). Liu even provides an explicit formula for the difference between the

Artin conductor and both Δ_0 and $\Delta_{\min}$ that can be described in terms of the combinatorics of the special fiber of the minimal regular model (of which there are already over 120 types!). This leads one to ask the following question, which we do not address in this paper.

Question 1.2. Can one give an interpretation of the difference between $\operatorname{Art}(X/K)$ and $\Delta_{X/K}$ in Theorem 1.1, analogous to the interpretation given in [Liu94]?

The Artin conductor for a regular model can be rewritten in terms of the number of components of the special fiber of the model as follows.

Proposition 1.3 ([Liu94, Proposition 1]). *If $\mathcal{X}/\mathcal{O}_K$ is a regular model of X, then the Artin conductor $\operatorname{Art}(\mathcal{X}/\mathcal{O}_K)$ satisfies*

$$-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = n - 1 + \varphi,$$

where n is the number of irreducible components of $\mathcal{X}_s$ and φ is the conductor exponent for the Galois representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$), which only depends on X.

If $\mathcal{X}$ is a proper regular model of X, then the number of irreducible components of $\mathcal{X}_s$ is at least the number of irreducible components in the special fiber of the minimal regular model of the curve X. Since the minimal discriminant of a hyperelliptic curve X is equal to the discriminant of one of the integral polynomials f that defines it, Proposition 1.3 shows that it suffices to produce, for every integral polynomial $f \in \mathcal{O}_K[x]$, a proper regular model $\mathcal{X}_f$ of X such that

$$-\operatorname{Art}(\mathcal{X}_f/\mathcal{O}_K) \leq \nu_K(\operatorname{disc}'(f)). \tag{1.4}$$

We call (1.4) the *conductor-discriminant inequality for f*. We may further also assume that K is strictly Henselian, since the invariants in (1.4) are constant under unramified base change and regular models satisfy étale descent. In other words, we have the following proposition.

Proposition 1.5. *If the conductor-discriminant inequality (1.4) holds for all $f \in \mathcal{O}_K[x]$, where K is Henselian with algebraically closed residue field of characteristic not 2, then Theorem 1.1 is true.*

1.1. **Earlier work of the second author.** Assume for the rest of the introduction that $\deg(f)$ is even, so $\operatorname{disc}'(f) = \operatorname{disc}(f)$. The common technique of [Sri15], [Sri19], and this paper is to produce an explicit regular model $\mathcal{X}_f$ admitting a finite degree 2 map to an explicit regular model $\mathcal{Y}_f$ of $\mathbb{P}^1_K$. The model $\mathcal{Y}_f$ is an embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, B)$, where B is the branch locus of the normalization of the standard model $\mathbb{P}^1_{\mathcal{O}_K}$ in $K(X)$. That is, $\mathcal{Y}_f$ is a blowup of $\mathbb{P}^1_{\mathcal{O}_K}$ on which all components of $\operatorname{div}(f)$ of odd multiplicity are regular and disjoint. In [Sri15], the assumption that the roots of f are defined over K ensures that all irreducible components of $\operatorname{div}(f)$ are already regular in the standard model $\mathbb{P}^1_{\mathcal{O}_K}$, and we only have to deal with separating the components of the branch divisor. The conductor-discriminant inequality for f is then proven by decomposing both $-\operatorname{Art}(\mathcal{X}_f)$ and $\nu_K(\operatorname{disc}(f))$ into local terms indexed by the vertices of the dual tree of $\mathcal{Y}_f$. When the roots of f are not defined over K, this analysis is much more involved, since we now need to carry out explicit embedded resolution of $\operatorname{div}(f)$ in $\mathbb{P}^1_{\mathcal{O}_K}$. In [Sri19], the dual tree of $\mathcal{Y}_f$ is enriched to a "metric tree", with lengths attached to the edges of the tree. This metric tree can be viewed as a combinatorial refinement of the discriminant and is built out of explicit Newton-Puiseux expansions of the roots of f, and can be used to gain traction on the explicit embedded resolution $\mathcal{Y}_f$. The proof of the conductor-discriminant inequality in [Sri19] is an induction on this metric tree, and makes use of refinements of tools introduced by Abhyankar to study resolutions of plane curve singularities. When the roots of f are no longer defined over a tame extension, this strategy breaks down since Newton-Puiseux expansions of the roots of f can have unbounded denominators and are hard to work with directly.

1.2. **Basic outline of the argument.** The key innovation in this paper is to use so-called "Mac Lane valuations" to describe the embedded resolution $\mathcal{Y}_f$ from §1.1. Mac Lane valuations are certain discrete valuations on the function field of $K(x)$ that were written down over 80 years ago by Mac Lane ([Mac36]) in a very convenient and useful notation. These valuations correspond to irreducible components on the special fibers of normal models of $\mathbb{P}^1_K$, and they allow us to discuss these models explicitly without getting into details about coordinate charts. In particular, they let us directly describe the regular model $\mathcal{Y}_f$, without having to go through expansions of roots of f. This has the added bonus that we need not split our proof into "tame" and "wild" cases. Additionally, using Mac Lane valuations to describe $\mathcal{Y}_f$ allows us to work over K for the entire paper — we do not pass to an extension L/K where X has semi-stable reduction, quotient by the Galois action, and resolve singularities. This is critical, since in general L/K could be extremely wildly ramified, which would force us to resolve quotient singularities with arbitrarily complicated inertia groups.

Now, the theory of Mac Lane valuations specifically concerns rational function fields rather than hyperelliptic ones. In light of this, as a first step in our proof, we reduce the proof of the conductor-discriminant inequality to an inequality between the number of components of the model $\mathcal{Y}_f$ and the "discriminant bonus"

$$\mathrm{db}_K(f) := \nu_K(\mathrm{disc}(f)) - \sum_{i=1}^{r} \nu_K(\mathrm{disc}(K_i/K)), \tag{1.6}$$

where $f = f_1 \cdots f_r$ is an irreducible factorization in $K[x]$ and K_i is the field generated by a root of f_i. Namely, Remark 2.9 says that $-\mathrm{Art}(\mathcal{X}_f/\mathcal{O}_K) \leq \nu_K(\mathrm{disc}(f))$ if and only if

$$2(N_{\mathcal{Y}_f,\mathrm{even}} - 1) \leq \mathrm{db}_K(f), \tag{1.7}$$

where $N_{\mathcal{Y}_f,\mathrm{even}}$ is the number of irreducible components of the special fiber of $\mathcal{Y}_f$ on which the order of f is even (see Proposition 2.8). This allows us to spend virtually the entire paper discussing models of $\mathbb{P}^1_K$ — we rarely deal with the hyperelliptic curve as such.

Our explicit description of $\mathcal{Y}_f$ using Mac Lane valuations lets us get good upper bounds on the number of components in its special fiber (see Theorem 8.1 and Corollary 10.18). The complications arising from wild ramification show up while trying to get explicit lower bounds on the discriminant bonus (see Proposition 7.31). Once we have both of these bounds, we can compare them to prove (1.6), and thus the conductor-discriminant inequality.

When the residue characteristic is 2, the branch locus of the double cover $\mathcal{X}_f \to \mathcal{Y}_f$ is not simply the components of odd multiplicity in $\mathrm{div}(f)$. So it is not straightforward to construct $\mathcal{Y}_f$, nor is it easy to compute the Swan conductor. However, work of Lorenzini and Liu ([LL99]) shows that one may still write down a regular model of X by taking the double cover of a well-chosen regular model of $\mathbb{P}^1_K$. So one might still hope to describe the model $\mathcal{Y}_f$ explicitly using Mac Lane valuations and prove that the conductor-discriminant inequality holds in residue characteristic 2 as well.

1.3. **Related work.** Several people have worked on comparing conductor exponents and discriminants. In the semistable case, work of Kausz [Kau99] (when $p \neq 2$) and Maugeais [Mau03] (all p) compares the Artin conductor to yet another notion of discriminant. In [DDMM18], the authors compute many arithmetic invariants attached to hyperelliptic curves in the semistable case in terms of the cluster picture of the polynomial f (which encodes the same information as the metric tree). In [Koh19], Kohls compares the conductor exponent φ with the minimal discriminant of superelliptic curves, by studying the Galois action on the special fiber of the semistable model as in [BW17]. In [BKSW19], the authors define minimal discriminants of Picard curves (degree 3 cyclic covers of $\mathbb{P}^1_K$) and compare the conductor exponent and the minimal discriminant for such curves.

There has also been a recent flurry of activity in constructing regular models of curves. In [NF19], Faraggi and Nowell describe the special fibers of SNC models of hyperelliptic curves when the splitting field of f is tamely ramified by resolving tame quotient singularities of the quotient of the semistable model (which they explicitly describe using the cluster picture/metric tree) by the Galois action. We cannot directly use their constructions, since the conductor-discriminant inequality does not hold with the minimal snc-model in place of the minimal regular model $\mathcal{X}_f$, as can already be seen in genus 1 when the minimal regular model does not coincide with the minimal snc-model. See also Example 1.8.

Parts of [NF19] rely on a very general result of Dokchitser ([Dok18]), which outlines a toric approach for constructing regular models of "sufficiently generic" curves from the Newton polygon of a defining equation. This approach can sometimes handle wild ramification, but doesn't include all hyperelliptic curves. Indeed, even in the tame case, the only hyperelliptic curves that [NF19] can directly apply [Dok18] to are those with "nested cluster pictures", i.e., those for which it is not possible to find two disjoint non-archimedean disks in $\mathbb{P}^1_{\overline{K}}$, each containing at least 2 roots of f (see [NF19, Definition 2.11, beginning of §4]). The hyperelliptic curve in Example 1.10 fails this criterion, as the roots of f can be placed into 3 disks, each containing two roots.

1.4. **Outline of the paper.** We now give a more detailed outline of the paper and our argument. Suppose we have a hyperelliptic curve given by an equation $y^2 = f(x)$ over K, where for simplicity $f(x) \in \mathcal{O}_K[x]$ is monic of even degree. In §2 we define the discriminant bonus of the polynomial f (cf. (1.6)). As was mentioned in §1.2, one can rephrase the conductor-discriminant inequality as an inequality involving the discriminant bonus of f and the number of irreducible components of a regular model of $\mathbb{P}^1_K$ whose normalization in the field extension $K(x)[\sqrt{f}]/K(x)$ is regular (Remark 2.9). After this rephrasing, we no longer need to deal with the Swan conductor directly, nor even the hyperelliptic curve itself.

In §3 we reduce to the case where all roots of f have positive valuation. In §4, we introduce Mac Lane valuations. As we have mentioned, a normal model of $\mathbb{P}^1_K$ corresponds to a finite set of Mac Lane valuations, one valuation for each irreducible component of the special fiber. Mac Lane valuations are also in one-to-one correspondence with *diskoids*, which are Galois orbits of rigid-analytic disks in $\mathbb{P}^1_{\overline{K}}$. We will use the diskoid perspective often, and it is introduced in in §4.2.

In §5, we prove several results about the correspondence between Mac Lane valuations and normal models of $\mathbb{P}^1_K$. For instance, if $\mathcal{Y}$ is a normal model of $\mathbb{P}^1_K$ with special fiber consisting of several irreducible components, each corresponding to a Mac Lane valuation, results in §5 can be used to determine which irreducible component a point of $\mathbb{P}^1_K$ specializes to. After this, we cite a result (Proposition 5.12) from [OW18] giving an explicit criterion for when a normal model of $\mathbb{P}^1_K$ is regular. More specifically, using that Mac Lane valuations correspond to normal models of $\mathbb{P}^1_K$ wth irreducible special fiber, Proposition 5.12 takes a Mac Lane valuation as input and gives the minimal regular resolution of the corresponding normal model as output (as a finite set of Mac Lane valuations, of course)!

The proof begins in earnest in §6, where our first major result, Theorem 6.9, is proved. Theorem 6.9 answers the question: given a monic irreducible polynomial $f \in \mathcal{O}_K[x]$, how do we construct a regular model of $\mathbb{P}^1_K$ on which the horizontal part of $\operatorname{div}(f)$ is regular? It turns out that one can associate a Mac Lane valuation v_f to such an f — this is the unique Mac Lane valuation over which f is a so-called "proper key polynomial" (see §4). After modifying v_f slightly to form a new Mac Lane valuation v'_f, we form the normal model of $\mathbb{P}^1_K$ whose irreducible special fiber corresponds to v'_f. Theorem 6.9 says that the horizontal part of $\operatorname{div}_0(f)$ is regular on the minimal regular resolution $\mathcal{Y}'_f$ of this model. Since $\mathcal{Y}'_f$ can be computed in terms of the Mac Lane valuation v_f, we conclude §6 by computing an upper bound on the number of its components in terms of f. The vertical components of $\operatorname{div}(f)$ are all isomorphic to $\mathbb{P}^1_k$, so at this point our model $\mathcal{Y}'_f$ has the property that its normalization in $K(x)[\sqrt{f}]$ has branch locus consisting of regular irreducible components (however, they might not be disjoint).

In §7, which has a more number-theoretic flavor than the rest of the paper, we turn our attention to the discriminant bonus of an irreducible monic polynomial $f \in \mathcal{O}_K[x]$ — we compute a lower bound on the discriminant bonus of f in terms of the Mac Lane valuation v_f from §6 (Proposition 7.31). This Mac Lane valuation is written as follows:

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n],$$

where the φ_i are polynomials of increasing degree and the λ_i are increasing positive rational numbers. Without giving any definitions, let us mention that we should think of the roots of the φ_i as being successively better approximations to the roots of f, defined over extensions of K of successively larger degree. The λ_i measure how close the roots of φ_i are to the roots of f (higher is closer). Having large values of n and λ_i indicates that roots of f have good lower degree approximations, and this places upward pressure on the discriminant bonus of f. It is here that complications related to allowing the splitting field of f to be wildly ramified over K show up, ultimately because knowing the degree of two wildly ramified extensions of K is not sufficient to determine whether one is contained in the other.

On the other hand, large values of n and the λ_i also give rise to more irreducible components on the special fiber of $\mathcal{Y}'_f$, as is computed in §6. In the short §8, we compare the number of these components from the calculation in §6 to the discriminant bonus of f calculated in §7. In §9, we make additional modifications to $\mathcal{Y}'_f$ to force the irreducible components of the branch locus of its normalization in $K(x)[\sqrt{f}]$ to be disjoint, and thus for the resulting normal model of the hyperelliptic curve to be regular. Combining a count of how many blowups are required to do this with the inequality from §8 proves the conductor-discriminant inequality when f is irreducible.

When f is reducible, the analogous computation is much more difficult, as we must separate the horizontal divisors of the various irreducible factors of f. It turns out that the number of blowups necessary to do this can also be bounded from above using Mac Lane valuations, and this bound can be expressed in terms of valuations of the resultants of pairs of the irreducible factors of f. This calculation is the subject of §10. Since these resultant valuations figure into $\operatorname{disc}(f)$, and thus into the discriminant bonus of f as well, this allows us to prove the conductor-discriminant inequality when f is reducible.

In Appendix A, we justify some of the constructions in the examples below.

1.5. **Examples.** We give several illustrative examples of the conductor-discriminant inequality. In all examples, it is assumed that k is algebraically closed and $\operatorname{char} k \neq 2$.

Example 1.8. Consider the hyperelliptic curve X given by the affine equation $y^2 = f(x)$, where $f(x) = x^d - \pi_K$ and π_K is a uniformizer of K. In this case, the normalization $\mathcal{X}$ of $\mathbb{P}^1_{\mathcal{O}_K}$ (with coordinate x) in the function field $K(X)$ is already regular.

Assume d is even for simplicity. Then $\chi(\mathcal{X}_{\overline{K}}) = 4 - d$. On the other hand, the special fiber of $\mathcal{X}$ is given by the affine equation $y^2 = x^d$, so it is a union of two copies of $\mathbb{P}^1_k$ meeting at one point. Thus $\chi(\mathcal{X}_s) = 2 - 0 + 1 = 3$. So $-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = d - 1 + \delta$, where δ is the Swan conductor. Using, e.g., Proposition 2.3, one calculates $\delta = \nu_K(d)$. We also have $\nu_K(\operatorname{disc}(f)) = \nu_K(d) + d - 1$. Thus the conductor-discriminant inequality is an equality in this case.

Note that the special fiber of $\mathcal{X}$ does *not* have simple normal crossings when $d \geq 4$, since the irreducible components do not meet transversely. By Proposition 1.3, the minimal snc-model $\mathcal{X}'$ of X has $-\operatorname{Art}(\mathcal{X}'/\mathcal{O}_K) > -\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = \operatorname{disc}(f)$, which means that $\mathcal{X}'$ does *not* satisfy the conductor-discriminant inequality. So minimal snc-models are insufficient for our purposes.

For the next two examples, the regular models are constructed using Mac Lane valuations. For details and justification of the constructions, see Appendix A. In both examples, we use the reformulation (1.7) of the conductor-discriminant inequality.

Example 1.9. The conductor-discriminant equality is not always an equality. Examples of strict inequality have been constructed, e.g., in [Liu94], [Sri15], and [Sri19]. We construct a simple one here. Consider the hyperelliptic curve X given by $y^2 = f(x)$, where $f(x) = x^8 - \pi_K^3$. One can verify that this equation realizes the minimal discriminant of X.

There is a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ consisting of 4 irreducible components in a chain configuration satisfying the criteria above, so its normalization $\mathcal{X}$ in $K(X)$ is a regular model of X (Figure 1). The order of f is even on 3 of these components, corresponding to X_0, X_2, and X_3 in Figure 1. On the other hand, if $L = K(\sqrt[8]{\pi_K})$ is the field generated by a root of f, then $\nu_K(\operatorname{disc}(L/K)) = 7$ and $\nu_K(\operatorname{disc}(f)) = 21$, so $\operatorname{db}_K(f) = 14$. Since $14 > 2(3-1)$, the reformulation (1.7) of the conductor-discriminant inequality is strict for $\mathcal{X}$.

In fact, $\mathcal{X}$ is not the minimal regular model of X, so the conductor-discriminant inequality for the minimal model $\mathcal{X}_{\min}$ is even stricter. For further detail, see Example A.1.

Example 1.10. We give an example where the minimal regular model fundamentally changes when the splitting field of $f(x)$ is wildly ramified over K. Consider the hyperelliptic curve X given by $y^2 = f(x)$, where $f(x)$ is the minimal polynomial of $\pi_K^{1/3} + \pi_K^{1/2}$. First, suppose $\operatorname{char} k \neq 3$. There is a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ consisting of 4 components in a chain

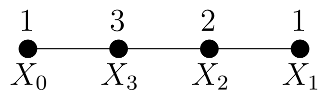

FIGURE 1. The dual graph of the model $\mathcal{Y}$ in Example 1.9. Components X_i are labeled above with their multiplicities. The X_i are listed in the order that they appear as exceptional divisors of point blowups of $\mathbb{P}^1_{\mathcal{O}_K}$, whose special fiber has strict transform X_0.

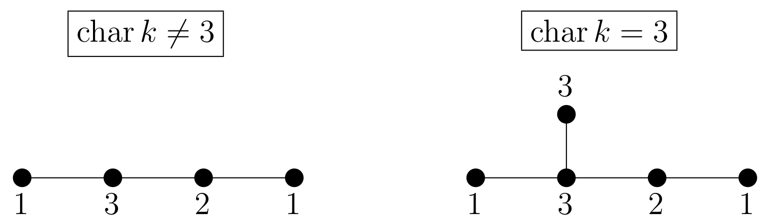

FIGURE 2. The dual graph of the model $\mathcal{Y}$ in Example 1.10 where $\operatorname{char} k \neq 3$ and $\operatorname{char} k = 3$. Components are labeled with their multiplicities.

configuration satisfying the criteria above (Figure 2), so its normalization $\mathcal{X}$ in $K(X)$ is a regular model of X. Furthermore, f has even order on all 4 components. Now, one calculates that $\nu_K(\operatorname{disc}_K(f)) = 11$, whereas $\nu_K(\operatorname{disc}_K(L/K)) = 5$, where $L = K[x]/(f) \cong K(\sqrt[6]{\pi_K})$. So $\operatorname{db}_K(f) = 11 - 5 = 6$. Since $N_{\mathcal{Y},\text{even}} = 4$, we get the equality $6 = 6$ in (1.7), and the conductor-discriminant inequality is an equality.

Now, suppose that $\operatorname{char} k = 3$, for example, say $K = \mathbb{Q}_3^{ur}$. The smallest regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ satisfying the criteria above now has 5 irreducible components arranged as in Figure 2, and f again has even order on all 5 components. But L/K is now wildly ramified, and this increases the discriminant bonus of f. Specifically, $\nu_K(\operatorname{disc}_K(f)) = 19$, whereas $\nu_K(\operatorname{disc}_K(L/K)) = 11$, so $\operatorname{db}_K(f) = 8$. Again, (1.7) is an equality.

In both cases, the normalization $\mathcal{X}$ of $\mathcal{Y}$ in $K(X)$ is in fact the minimal regular model of X. For further detail, see Example A.2.

NOTATION AND CONVENTIONS

Throughout, K is a strictly Henselian field with respect to a discrete valuation ν_K with residue characteristic not 2. We denote *fixed* separable and algebraic closures of K by $K^{\text{sep}} \subseteq \overline{K}$. All algebraic extensions of K are assumed to live inside $\overline{K}$. This means that for any algebraic extension L/K, there is a preferred embedding $\iota_L \in \operatorname{Hom}_K(L, \overline{K})$, namely the inclusion.

We fix a uniformizer π_K of ν_K and normalize ν_K so that $\nu_K(\pi_K) = 1$. We further assume that the residue field k of K is *algebraically closed*. If L/K is an algebraic extension, the valuation ν_K extends uniquely to L, and we write $\mathcal{O}_L$ (resp. π_L) for the valuation ring (resp. a uniformizer) of L. If L/K is a *finite* algebraic extension, we write ν_L for the renormalization of the extension of ν_K to L such that $\nu_L(\pi_L) = 1$. In fact, we conflate ν_K and ν_L with their extensions to $\overline{K}$, and furthermore with the restrictions of these extended valuations to subextensions of $\overline{K}/K$. So, for instance, if L and M are finite extensions of K and $m \in M$, it makes sense to evaluate $\nu_L(m)$, even if L and M are linearly disjoint over K. We simply have $\nu_L(m) = [L : K]\nu_K(m) = [L : K][M : K]^{-1}\nu_M(m)$.

For a finite separable field extension L/K, we let $\mathrm{disc}(L/K)$ denote the discriminant of the field extension L/K and let $\Delta_{L/K} := \nu_K(\mathrm{disc}(L/K))$. For any separable polynomial $f \in K[x]$, we let $\mathrm{disc}(f)$ (resp. $\mathrm{disc}'(f)$) denote the discriminant of the polynomial f viewed as a polynomial of degree $\deg(f)$ (resp. degree $2\lceil \deg(f)/2 \rceil$) and let $\Delta_{f,K} := \nu_K(\mathrm{disc}(f))$. Note that with this convention, if $f = cg$ for some monic polynomial g, then $\Delta_{f,K} = 2\nu_K(c)(\deg(g) - 1) + \Delta_{g,K}$. If f_i and f_j are a pair of polynomials in $K[x]$ with no common roots, we denote their resultant by $\mathrm{Res}(f_i, f_j)$ and let $\rho_{f_i,f_j,K} := \nu_K(\mathrm{Res}(f_i, f_j))$. We will suppress the index K whenever the field is clear.

All minimal polynomials are assumed to be monic. The *K-degree of an element* $\alpha \in \overline{K}$ is the degree of its minimal polynomial over K. When we refer to the *denominator* of a rational number, we mean the positive denominator when the rational number is expressed as a reduced fraction.

For an integral K-scheme or $\mathcal{O}_K$-scheme S, we denote the corresponding function field by $K(S)$. If $\mathcal{Y} \to \mathcal{O}_K$ is an arithmetic surface, an irreducible codimension 1 subscheme of $\mathcal{Y}$ is called *vertical* if it lies in a fiber of $\mathcal{Y} \to \mathcal{O}_K$, and *horizontal* otherwise. Let $f \in K(\mathcal{Y})$. We denote the divisor of zeroes of f by $\mathrm{div}_0(f)$. If $\mathrm{div}(f) = \sum m_i \Gamma_i$, call a component Γ_i for which m_i is odd an *odd component* of $\mathrm{div}(f)$ on $\mathcal{Y}$. Similarly define *even component* of $\mathrm{div}(f)$ (this includes every component Γ_i for which $m_i = 0$). For any discrete valuation v, we denote the corresponding value group by Γ_v. If E is a regular codimension 1 point of $\mathcal{Y}$, we will denote the corresponding discrete valuation on the function field of $\mathcal{Y}$ by ν_E. If P is a closed point on $\mathcal{Y}$, we denote the corresponding local ring by $\mathcal{O}_{\mathcal{Y},P}$ and maximal ideal by $\mathfrak{m}_{\mathcal{Y},P}$.

Throughout this paper, we will let $X \to \mathbb{P}^1_K$ be a hyperelliptic curve over K of genus $g \geq 1$. Write $\mathbb{P}^1_K = \mathrm{Proj}\, K[x_0, x_1]$ and let $x := x_1/x_0$. Let $\mathbb{P}^1_{\mathcal{O}_K} := \mathrm{Proj}\, \mathcal{O}_K[x_0, x_1]$. We assume that X is given by an (affine) equation $y^2 = f$, with $f \in K[x]$.

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2. The discriminant bonus and regular models

Recall that X/K is a hyperelliptic curve with affine equation $y^2 = f(x)$. The *discriminant* of such an equation is the integer $\nu_K(\mathrm{disc}'(f))$. Changing x-coordinates on $\mathbb{P}^1_K$ using an element of $\mathrm{GL}_2(\mathcal{O}_K)$ does not change the valuation of the discriminant of an equation. Since k is algebraically closed, we may assume that f has even degree by such a change of coordinates [Sri15, Section 1.3], and we may even assume that no root of f specializes to ∞. That is, we may assume that all roots of f lie in $\mathcal{O}_K$. If f has repeated roots, then $\mathrm{disc}'(f) = 0$ and (1.4) is satisfied automatically, so assume also that f is separable. Lastly, since K is Henselian with algebraically closed residue field of characteristic not 2, the group

$K^\times/(K^\times)^2$ has two elements whose coset representatives are 1 and π_K. So after multiplying f by squares, which does not change the isomorphism class of X, we may assume that the leading coefficient of f is 1 or π_K. Thus, for the remainder of the paper, we make the following assumption:

Assumption 2.1. The polynomial $f(x)$ has even degree, is separable, and has irreducible factorization $\pi_K^b f_1 \dots f_r$, where the $f_i \in \mathcal{O}_K[x]$ are monic irreducible polynomials and $b \in \{0, 1\}$.

The argument above proves the following proposition.

Proposition 2.2. *If the conductor-discriminant inequality holds for all f satisfying Assumption 2.1, then it holds for all $f \in \mathcal{O}_K[x]$.*

For f satisfying Assumption 2.1, we define $K_i = K[x]/f_i(x)$ for $1 \leq i \leq r$.

Proposition 2.3. *The Swan conductor of X equals $\sum_{i=1}^r (\Delta_{K_i/K} - \deg f_i + 1) = r - \deg(f) + \sum_{i=1}^r \Delta_{K_i/K}$.*

Proof. The argument in [DDMM18, Theorem 1.20(i)] for K a local field works also for K Henselian discretely valued with algebraically closed residue field, with the added simplification that all residue degrees are 1. □

Definition 2.4. The *discriminant bonus of f over K*, written $\mathrm{db}_K(f)$, is the quantity $\Delta_{f,K} - \sum_{i=1}^r \Delta_{K_i/K}$. If $\alpha \in \overline{K}$, we define its discriminant bonus $\mathrm{db}_K(\alpha)$ over K to be $\mathrm{db}_K(f)$, where f is the minimal polynomial for α over K.

Remark 2.5. If $f = \pi_K^b g_1 g_2 \dots g_r \in \mathcal{O}_K[x]$ is any factorization of f where $b \in \{0, 1\}$ and the $g_i \in \mathcal{O}_K[x]$ are monic, not necessarily irreducible polynomials, then using properties of discriminants and resultants, we have

$$\Delta_{f,K} = \sum \Delta_{g_i,K} + \sum_{1 \leq i < j \leq r} 2\rho_{g_i,g_j,K} + 2b(\deg(f) - 1).$$

Using this and Definition 2.4, we get

$$\mathrm{db}_K(f) = \sum_{i=1}^r \mathrm{db}_K(g_i) + \sum_{1 \leq i < j \leq r} 2\rho_{g_i,g_j,K} + 2b(\deg(f) - 1). \tag{2.6}$$

The following estimate will be used in §10.

Lemma 2.7. *Let $f_1, f_2 \in \mathcal{O}_K[x]$ be monic and irreducible, such that all roots of f_1 and f_2 have positive valuation. Let α_1 and α_2 be roots of f_1 and f_2 respectively such that $\nu_K(\alpha_1 - \alpha_2)$ is minimal.*

(i) *Unless one of f_1 or f_2 is Eisenstein and the other is linear, $\rho_{f_1,f_2} \geq 2$.*
(ii) *Suppose there exists $\gamma \in K$ such that $\nu_K(\alpha_1 - \gamma) = a/2$ with $a \geq 3$ an odd integer, and f_2 linear with $\nu_K(\alpha_2 - \gamma) > a/2$. Then $\rho_{f_1,f_2} \geq 3$.*
(iii) *If f_1 is non-Eisenstein with $\deg(f_1) \geq 3$ and f_2 is Eisenstein with $\deg(f_2) \geq 2$, then $\rho_{f_1,f_2} \geq 3$.*
(iv) *If neither f_1 nor f_2 is Eisenstein or linear, then $\rho_{f_1,f_2} \geq 4$.*

Proof. For part (i), assume without loss of generality that $\nu_K(\alpha_1) \leq \nu_K(\alpha_2)$. Then $\nu_K(\alpha_1 - \alpha_2) \geq \nu_K(\alpha_1)$, and the same is true whenever α_1 or α_2 is replaced by one of its K-conjugates. So $\rho_{f_1,f_2} \geq \deg(f_1)\deg(f_2)\nu_K(\alpha_1)$. Now, since we assumed all roots of f_1 and f_2 have strictly positive valuation, $\nu_K(\alpha_1) \geq 1/\deg(f_1)$ with equality holding only if α_1 is Eisenstein, and $\deg(f_2) \geq 1$ with equality holding only if f_2 is linear. So $\rho_{f_1,f_2} \geq 1$, with equality only if α_1 is Eisenstein and f_2 is linear. This proves part (i).

The inequalities in the remaining three parts use the following estimate, which follows from the fact that we picked α_1, α_2 to be roots of f_1 and f_2 such that $\nu_K(\alpha_1 - \alpha_2)$ is minimal:

$$\rho_{f_1,f_2} \geq \deg(f_1)\deg(f_2)\nu_K(\alpha_1 - \alpha_2).$$

For part (ii), since we assumed that $\nu_K(\alpha_1 - \gamma) = a/2$ and $\nu_K(\alpha_2 - \gamma) > a/2$, we have $\nu_K(\alpha_1 - \alpha_2) = a/2$. Further since $\gamma \in K$ and a is odd, we have $\deg(f_1) \geq 2$. Putting these together, we get $\rho_{f_1,f_2} \geq 2(1)(a/2) = a \geq 3$. For part (iii), $\nu_K(\alpha_1 - \alpha_2) \geq \min(2/\deg(f_1), 1/\deg(f_2))$, so $\rho_{f_1,f_2} \geq \min(2\deg(f_2), \deg(f_1)) \geq 3$, so part (iii) follows. In the situation of part (iv), we have $\nu_K(\alpha_1 - \alpha_2) \geq \min(2/\deg(f_1), 2/\deg(f_2))$, so $\rho_{f_1,f_2} \geq \min(2\deg(f_2), 2\deg(f_1))$, and part (iv) follows. ◻

We now obtain a formula for the Artin conductor. Let $\mathcal{Y}$ be a regular model of $\mathbb{P}^1_K$ and let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $K(x)[y]/(y^2 - f(x))$. Let B be the branch locus of $\mathcal{X} \to \mathcal{Y}$. Write $\mathcal{Y}_s$ and $\mathcal{Y}_{\overline{\eta}} \cong \mathbb{P}^1_{\overline{K}}$ for the special and geometric generic fibers of $\mathcal{Y}$, respectively, and write B_s and $B_{\overline{\eta}}$ for the special and geometric generic fibers of B, respectively. Let $N_{\mathcal{Y}}$ be the number of irreducible components of $\mathcal{Y}_s$, and let $N_{\mathcal{Y},\text{odd}}/N_{\mathcal{Y},\text{even}}$ be the number of odd/even vertical components of $\text{div}(f)$.

Proposition 2.8. *Keep the notation from the paragraph above. Assume that the odd components of* $\text{div}_0(f)$ *are regular and pairwise disjoint. Then* $\mathcal{X}$ *is regular and we have*

$$-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = 2(N_{\mathcal{Y}} - 1) - 2N_{\mathcal{Y},\text{odd}} + \sum_{i=1}^{r} \Delta_{K_i/K} = 2(N_{\mathcal{Y},\text{even}} - 1) + \sum_{i=1}^{r} \Delta_{K_i/K}$$

Proof. By [Sri15, Lemma 2.1], the model $\mathcal{X}$ is regular. By [Sri19, Lemma 2.2], we have

$$-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = 2(\chi(\mathcal{Y}_s) - \chi(\mathcal{Y}_{\overline{\eta}})) - (\chi(B_s) - \chi(B_{\overline{\eta}})) + \delta,$$

where δ is the Swan conductor of X. We will use H^i and h^i to denote the étale cohomology groups and their dimensions respectively. Now, $\mathcal{Y}_s$ and $\mathcal{Y}_{\overline{\eta}}$ both have trivial H^1 and one-dimensional H^0, while $h^2(\mathcal{Y}_s) = N_{\mathcal{Y}}$ and $h^2(\mathcal{Y}_{\overline{\eta}}) = 1$. So $\chi(\mathcal{Y}_s) - \chi(\mathcal{Y}_{\overline{\eta}}) = N_{\mathcal{Y}} - 1$. Since $\deg(f)$ is even by Assumption 2.1, it follows that B consists of precisely all the odd components of $\text{div}_0(f)$. Since the odd components of $\text{div}_0(f)$ are regular and pairwise disjoint, it follows that as a closed subset, B_s is a disjoint union of closed points and closed codimension 1 sets: the closed points correspond to points where the horizontal components of $\text{div}_0(f)$ specialize, so there is exactly one for each irreducible factor of f, and the codimension 1 sets correspond to the vertical components appearing with odd multiplicity in $\text{div}(f)$ on $\mathcal{Y}$. By [OW18, Lemma 7.1], these irreducible components are all isomorphic to $\mathbb{P}^1_k$ and therefore have trivial H^1, and since χ is an additive functor, it follows that $\chi(B_s) = r + 2N_{\mathcal{Y},\text{odd}}$. Since $\deg(f)$ is even, $B_{\overline{\eta}}$ consists of $\deg(f)$ points and therefore $\chi(B_{\overline{\eta}}) = \deg(f)$. Lastly, by Proposition 2.3, $\delta = r - \deg(f) + \sum_{i=1}^{r} \Delta_{K_i/K}$. Putting everything together proves the proposition. ◻

Remark 2.9. In light of Definition 2.4 of the discriminant bonus, in order to prove the conductor-discriminant inequality for f, it suffices to find a model $\mathcal{Y}$ of $\mathbb{P}^1_K$ satisfying the hypotheses of Proposition 2.8, such that

$$\mathrm{db}_K(f) \geq 2(N_{\mathcal{Y},\mathrm{even}} - 1). \tag{2.10}$$

We say that such a model $\mathcal{Y}$ *realizes the conductor-discriminant inequality for f.*

We also prove an algebro-geometric lemma which will be useful in §9 and §10.

Lemma 2.11. *Let S be a regular arithmetic surface, and let $P \in S$ be a closed point. Let $f \in K(S)$ be a rational function regular at P, and let $f = ug_1^{m_1} \dots g_r^{m_r}$ be an irreducible factorization of f in the local ring $\mathcal{O}_{S,P}$, where u is a unit and the g_i are pairwise distinct irreducible elements. Assume that P is a regular point of* $\mathrm{div}(g_i)$ *for every i. Let E be the exceptional curve for the blowup of S at P. Then $\nu_E(f) = \sum_{i=1}^r m_i$.*

Proof. Recall that the multiplicity $\mu_P(f)$ is defined to be the largest integer r such that $f \in \mathfrak{m}_{S,P}^r$. By [Liu02, Chapter 9, Proposition 2.23] we have $\nu_E(f) = \mu_P(f)$. The factorization $f = g_1^{m_1} \dots g_r^{m_r}$ implies $\mu_P(f) = \sum_{i=1}^r m_i \mu_P(g_i)$. Since P is assumed to be a regular point of $\mathrm{div}(g_i)$, by [Liu02, Chapter 9, Remark 2.20] we have $\mu_P(g_i) = 1$. Putting the last three sentences together proves the lemma. □

Remark 2.12. In the lemma above, if S is a model of $\mathbb{P}^1_K$ and $\mathrm{div}(g_i)$ is an irreducible vertical divisor, then $\mathrm{div}(g_i) \cong \mathbb{P}^1_k$ is regular (see for instance [OW18, Lemma 7.1]).

3. A key reduction step

The goal of this section is to show that we may assume all roots of f have strictly positive valuation in Remark 2.9 without any loss of generality. Let $f \in \mathcal{O}_K[x]$ be a polynomial satisfying Assumption 2.1 with leading coefficient π_K^b, where $b \in \{0, 1\}$. Let $f = \pi_K^b g_1 g_2 \dots g_s$ be a factorization of f where the roots of the $g_i \in \mathcal{O}_K[x]$ have pairwise distinct specializations in $\mathbb{P}^1_k$, and where all roots of a given g_i have the same specialization in $\mathbb{P}^1_k$. For each i, let $h_i := \pi_K^b g_i$. (Note that $\deg(h_i)$ might be odd.) For each i, let $\mathcal{Y}_i$ be a regular model of $\mathbb{P}^1_K$ equipped with a dominating map $\pi_i \colon \mathcal{Y}_i \to \mathbb{P}^1_{\mathcal{O}_K}$. Let P_i be the point where $\mathrm{div}_0(g_i)$ intersects the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$. Note that any model $\mathcal{Y}$ of $\mathbb{P}^1_K$ dominating all the $\mathcal{Y}_i$ also dominates $\mathbb{P}^1_{\mathcal{O}_K}$.

Lemma 3.1. *Let $f, h_i, P_i, \mathcal{Y}_i$ be as in the paragraph above. Assume that for each i,*

- *the base locus of $\pi_i \colon \mathcal{Y}_i \to \mathbb{P}^1_{\mathcal{O}_K}$ is either empty or $\{P_i\}$, and,*
- *if $\deg(h_i) = 1$, then $\mathcal{Y}_i = \mathbb{P}^1_{\mathcal{O}_K}$ (empty base locus for π_i) when $b = 0$ and $\mathcal{Y}_i$ consists of a single blowup at the closed point P_i when $b = 1$.*

Let $\mathcal{Y}_f$ be the minimal regular model of $\mathbb{P}^1_K$ that dominates all the $\mathcal{Y}_i$, and let $\pi \colon \mathcal{Y}_f \to \mathbb{P}^1_{\mathcal{O}_K}$ be the unique map factoring through the π_i. Let Γ be the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$ in $\mathcal{Y}_f$. Then we have the following.

(i) *Γ is an odd component of $\mathrm{div}(f)$ if and only if $b = 1$ if and only if Γ is an odd component of $\mathrm{div}(h_i)$ for every i.*

(ii) *If $b = \deg(h_i) = 1$, then the single exceptional curve above P_i in $\mathcal{Y}_i$ is an even component of $\mathrm{div}(h_i)$.*

(iii) *The special fiber of the model $\mathcal{Y}_f$ is obtained by gluing together the $\pi_i^{-1}(P_i)$ for various i to the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$.*

(iv) *For every i, under the identification from part (iii), we have*

$$\operatorname{div}(f)|_{\pi^{-1}(P_i)} = \operatorname{div}(h_i)|_{\pi^{-1}(P_i)} = \operatorname{div}(h_i)|_{\pi_i^{-1}(P_i)}.$$

(v) *If $\mathcal{Y}_i$ satisfies the hypotheses of Proposition 2.8 for every h_i with $\deg(h_i) \geq 2$, then $\mathcal{Y}_f$ satisfies the hypotheses of Proposition 2.8 for f.*

(vi) *Let $N_{\mathcal{Y}_f,\text{even}}$ denote the number of even components of $\operatorname{div}(f)$ in the special fiber of $\mathcal{Y}_f$. Let $N_{\mathcal{Y}_i,\text{even}}$ be the number of even components of $\operatorname{div}(h_i)$ in the special fiber of $\mathcal{Y}_i$. Then*

$$N_{\mathcal{Y}_f,\text{even}} - 1 = \sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}} - 1) + bs - b.$$

Proof.

(i) The order of vanishing of both f and h_i along Γ is b and $b \in \{0,1\}$.

(ii) When $b = 1$ and $\deg(h_i) = 1$, we have that $\operatorname{div}_0(h_i) = V + H$, where V and H are the vertical and horizontal irreducible components of $\operatorname{div}_0(h_i)$ meeting at P_i. Furthermore, P_i is a regular point of both V and H and by [Sri15, Lemma 2.3], a single blowup at P_i produces an even exceptional component E_i of $\operatorname{div}(h_i)$.

(iii) By assumption, the base locus of $\mathcal{Y}_i$ is a subset of $\{P_i\}$ and $P_i \neq P_j$ for $i \neq j$. Since blowups based at different points of $\mathbb{P}^1_{\mathcal{O}_K}$ can be performed in any order, and $\mathcal{Y}_f$ is the *minimal* model that dominates the $\mathcal{Y}_i$, it follows that $\mathcal{Y}_f$ is obtained by gluing together the $\pi_i^{-1}(P_i)$ for various i to the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$. In particular, $\pi^{-1}(P_i)$ and $\pi_i^{-1}(P_i)$ are isomorphic.

(iv) Since the $g_i \in \mathcal{O}_K[x]$ have pairwise distinct specializations in $\mathbb{P}^1_k$, we have $g_j \in \mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K},P_i} \setminus \mathfrak{m}_{\mathcal{O}_K,P_i}$ if $j \neq i$. It follows that h_i and f differ by a unit in $\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K},P_i}$, and therefore $\operatorname{div}_0(f)|_{\pi^{-1}(P_i)} = \operatorname{div}_0(h_i)|_{\pi^{-1}(P_i)} = \operatorname{div}_0(h_i)|_{\pi_i^{-1}(P_i)}$.

(v) We need to show that the odd components of $\operatorname{div}_0(f)$ are all regular and pairwise disjoint on $\mathcal{Y}_f$ assuming that the odd components of $\operatorname{div}_0(h_i)$ are all regular and pairwise disjoint on $\mathcal{Y}_i$ when $\deg(h_i) \geq 2$.

When $b = 1$ and $\deg(h_i) = 1$, we showed in part (ii) that a single blowup at P_i produces an even exceptional component E_i of $\operatorname{div}_0(h_i)$. Since E_i is even and regular, any closed point of E_i lies on at most one odd component of $\operatorname{div}_0(h_i)$. When $b = 0$ and $\deg(h_i) = 1$, the point P_i is a regular point of $\operatorname{div}_0(h_i)$ and $\operatorname{div}_0(h_i)$ is horizontal and irreducible. Thus, in both cases when $\deg(h_i) = 1$, the odd components of $\operatorname{div}_0(h_i)$ are regular and pairwise disjoint on $\mathcal{Y}_i$.

In the proof of part (iv), we saw that h_i and f differ by a unit in $\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K},P_i}$. It follows that the strict transforms of the odd horizontal components of $\operatorname{div}_0(f)$ in $\mathcal{Y}_f$ specializing to P_i are isomorphic to those of $\operatorname{div}_0(h_i)$ in $\mathcal{Y}_i$.

Note that every odd horizontal component of $\operatorname{div}_0(f)$ on $\mathbb{P}^1_{\mathcal{O}_K}$ specializes to one of the P_i. Combining this with the isomorphism $\pi^{-1}(P_i) \cong \pi_i^{-1}(P_i)$ and the assumption that odd horizontal components of $\operatorname{div}_0(h_i)$ are regular in $\mathcal{Y}_i$, pairwise disjoint and do not intersect an odd vertical component of $\operatorname{div}_0(h_i)$, we get that the odd horizontal components of $\operatorname{div}_0(f)$ are all regular, pairwise disjoint and do not intersect an odd vertical component of $\operatorname{div}_0(f)$. Since all vertical components of $\operatorname{div}_0(f)$ are isomorphic to $\mathbb{P}^1_k$ by [OW18, Lemma 7.1] and therefore regular, it remains to check that no two odd vertical components of $\operatorname{div}_0(f)$ intersect.

Since $\mathcal{Y}_f$ is obtained by gluing together the $\pi_i^{-1}(P_i)$ for various i to the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$, and we showed that $\operatorname{div}_0(f)|_{\pi^{-1}(P_i)} = \operatorname{div}_0(h_i)|_{\pi_i^{-1}(P_i)}$ it follows that the odd vertical components of $\operatorname{div}_0(f)$ supported in

$$\pi^{-1}(P_1) \sqcup \pi^{-1}(P_2) \sqcup \cdots \sqcup \pi^{-1}(P_s)$$

are pairwise disjoint. It remains to analyze the behaviour of $\operatorname{div}_0(f)$ along the strict transform Γ of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$ in $\mathcal{Y}_f$. When $b = 0$, since Γ is even by part (a), we are done. Now let $b = 1$. This makes Γ an odd component for $\operatorname{div}_0(f)$ on $\mathcal{Y}_f$ and also for $\operatorname{div}_0(h_i)$ on $\mathcal{Y}_i$. Since f and h_i differ by a unit in $\mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}, P_i}$, our hypothesis on $\mathcal{Y}_i$ implies that the component Γ does not meet any odd components of $\operatorname{div}_0(f)|_{\pi^{-1}(P_i)} = \operatorname{div}_0(h_i)|_{\pi_i^{-1}(P_i)}$ supported on $\pi^{-1}(P_i) = \pi_i^{-1}(P_i)$. This finishes the proof that no two odd vertical components of $\operatorname{div}_0(f)$ on $\mathcal{Y}_f$ intersect.

(vi) We analyze the two cases $b = 0$ and $b = 1$ separately, and use the identifications in parts (ii) and (iii) of the odd/even components in $\pi^{-1}(P_i)$ and $\pi_i^{-1}(P_i)$ to count the number of even components. First let $b = 0$. If $\deg(h_i) = 1$, then $\pi^{-1}(P_i) = \pi_i^{-1}(P_i) = \{P_i\}$ by assumption. By part (i), the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$ is an even component of $\operatorname{div}(f)$ in $\mathcal{Y}_f$ and also an even component of $\operatorname{div}(h_i)$ in $\mathcal{Y}_i$. For $\star \in \{f, i\}$, call a component of the special fiber of $\mathcal{Y}_\star$ that is not equal to the strict transform of $\mathbb{P}^1_{\mathcal{O}_K}$ a *new* component. By the previous two sentences, the number of new even components of $\mathcal{Y}_f$ is $N_{\mathcal{Y}_f,\text{even}} - 1$, and that of $\mathcal{Y}_i$ is $N_{\mathcal{Y}_i,\text{even}} - 1$. Using the previous line in the first and fourth equalities below, part (iii) in the second line, and part (i) and our assumption that $\mathcal{Y}_i = \mathbb{P}^1_{\mathcal{O}_K}$ when $b = 0$ and $\deg(h_i) = 1$ in the third line, we get

$$\begin{aligned} N_{\mathcal{Y}_f,\text{even}} - 1 &= \#\,(\text{new even components of } \mathcal{Y}_f) \\ &= \sum_i \#\left(\text{even components in } \pi_i^{-1}(P_i)\right) \\ &= \sum_{\deg(h_i)\geq 2} \#\left(\text{even components in } \pi_i^{-1}(P_i)\right) \\ &= \sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}} - 1) \\ &= \left(\sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}} - 1)\right) + bs - b. \end{aligned}$$

Now let $b = 1$. If $\deg(h_i) = 1$, then part (ii) shows that $\pi^{-1}(P_i) = \pi_i^{-1}(P_i)$ is a single even irreducible component, and therefore the number of even components in $\pi_i^{-1}(P_i)$ equals 1. Once again part (i) shows that the strict transform of the special fiber of $\mathbb{P}^1_{\mathcal{O}_K}$ is an odd component of $\operatorname{div}(f)$ in $\mathcal{Y}_f$ and also an odd component of $\operatorname{div}(h_i)$ in $\mathcal{Y}_i$. Combining this with part (iii) as before, and using parts (i) and (iii) in the first line, and $b = 1$ in the last line, we get

$$\begin{aligned}
N_{\mathcal{Y}_f,\text{even}} - 1 &= \left(\sum_i \# \left(\text{even components of } \pi_i^{-1}(P_i)\right) \right) - 1 \\
&= \left(\sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}}) + \sum_{\deg(h_i)=1} 1 \right) - 1 \\
&= \left(\sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}} - 1) \right) + \left(\sum_i 1 \right) - 1 \\
&= \left(\sum_{\deg(h_i)\geq 2} (N_{\mathcal{Y}_i,\text{even}} - 1) \right) + bs - b \qquad \square
\end{aligned}$$

Recall from Remark 2.9 what it means for a model $\mathcal{Y}$ of $\mathbb{P}^1_K$ to realize the conductor-discriminant inequality for a polynomial.

Proposition 3.2. *Let $f, h_i, \mathcal{Y}_i, \mathcal{Y}_f$ be as in the lemma above. Make the same three assumptions as in Lemma 3.1. Assume that $\mathcal{Y}_i$ satisfies the hypothesis of Proposition 2.8 for the polynomial h_i. Let $N_{\mathcal{Y}_i,\text{even}}$ be the number of even components of $\operatorname{div}(h_i)$ in the special fiber of $\mathcal{Y}_i$. If the inequality*

$$2(N_{\mathcal{Y}_i,\text{even}} - 1) \leq \operatorname{db}_K(h_i)$$

holds for every i with $\deg(h_i) \geq 2$, then $\mathcal{Y}_f$ realizes the conductor-discriminant inequality for f.

Proof. Since $h_i = \pi_K^b g_i$, by Remark 2.5 we have $\operatorname{db}_K(h_i) = \operatorname{db}_K(g_i) + 2b(\deg(g_i) - 1)$. Since the roots of the g_i have distinct specializations, we have $\rho_{g_i,g_j} = 0$ and since $f = u\pi_K^b g_1 \dots g_s$, by Remark 2.5 we have

$$\begin{aligned}
\operatorname{db}_K(f) &= \sum_{i=1}^{s} \operatorname{db}_K(g_i) + 2b(\deg(f) - 1) \\
&= \sum_{i=1}^{s} (\operatorname{db}_K(h_i) - 2b(\deg(g_i) - 1)) + 2b(\deg(f) - 1) \quad (\because h_i = \pi_K^b g_i) \\
&= \sum_{i=1}^{s} \operatorname{db}_K(h_i) + 2b(s - 1) \qquad\qquad (\because \deg(f) = \sum_{i=1}^{s} \deg(g_i)).
\end{aligned}$$

Let $N_{\mathcal{Y}_f,\text{even}}, N_{\mathcal{Y}_i,\text{even}}$ be as in Lemma 3.1(vi) above. By assumption $2(N_{\mathcal{Y}_i,\text{even}} - 1) \leq \operatorname{db}_K(h_i)$ for every i with $\deg(h_i) \geq 2$. Combining this with Lemma 3.1(vi), we get

$$2(N_{\mathcal{Y}_f,\text{even}} - 1) = \sum_{\deg(h_i)\geq 2} 2(N_{\mathcal{Y}_i,\text{even}} - 1) + 2b(s-1) \leq \sum_{\deg(h_i)\geq 2} \operatorname{db}_K(h_i) + 2b(s-1) = \operatorname{db}_K(f),$$

which shows that $\mathcal{Y}_f$ realizes the conductor-discriminant inequality for f. $\square$

Corollary 3.3. *If the conductor-discriminant inequality holds for every f satisfying Assumption 2.1 where the roots of f all have positive valuation, then the conductor-discriminant inequality holds for every f satisfying Assumption 2.1.*

Proof. Write $f = \pi_K^b g_1 \cdots g_s$ as in this section. By Proposition 3.2 together with Remark 2.9, it suffices to prove the conductor-discriminant inequality for each $h_i(x) = \pi_K^b g_i(x)$ separately. Let $\overline{a}_i \in k$ be the reduction of the roots of h_i (here we use that k is algebraically closed). If a_i is a lift of $\overline{a}_i$ to $\mathcal{O}_K$ and $\tilde{h}_i(x) = h_i(x + a_i)$, then the roots of $\tilde{h}_i(x)$ all have positive valuation, so by assumption the conductor-discriminant inequality holds for $\tilde{h}_i$. Now $\mathrm{disc}(h_i) = \mathrm{disc}(\tilde{h}_i)$, and the Artin conductor of minimal regular model of the hyperelliptic curve $y^2 = h_i$ equals that of $y^2 = \tilde{h}_i$ since the curves are isomorphic over K. Hence the conductor-discriminant inequality holds for h_i as well. □

In light of Corollary 3.3, the rest of the paper focuses on proving the conductor-discriminant inequality for polynomials f satisfying Assumption 2.1, all of whose roots have *positive valuation*.

4. Mac Lane valuations

4.1. **Definitions and facts.** We recall the theory of inductive valuations, which was first developed by Mac Lane in [Mac36]. Our main reference is [Rüt14]. Inductive valuations give us an explicit way to talk about normal models of $\mathbb{P}^1$, and will be essential for bounding the number of irreducible components of the special fiber of a regular model of a hyperelliptic curve.

Define a *geometric valuation* of $K(X)$ to be a discrete valuation that restricts to ν_K on K and whose residue field is a finitely generated extension of k with transcendence degree 1. We place a partial order $\preceq$ on valuations by defining $v \preceq w$ if $v(f) \leq w(f)$ for all $f \in K[x]$. Let v_0 be the *Gauss valuation* on $K(x)$. This is defined on $K[x]$ by $v_0(a_0 + a_1 x + \cdots a_n x^n) = \min_{0 \leq i \leq n} \nu_K(a_i)$, and then extended to $K(x)$.

We consider geometric valuations v such that $v \succeq v_0$. By the triangle inequality, these are precisely those geometric valuations for which $v(x) \geq 0$. This entails no loss of generality, since x can always be replaced by x^{-1}. We would like an explicit formula for describing geometric valuations, similar to the formula above for the Gauss valuation, and this is achieved by the so-called *inductive valuations* or *Mac Lane valuations*. Observe that the Gauss valuation is described using the x-adic expansion of a polynomial. The idea of a Mac Lane valuation is to "declare" certain polynomials φ_i to have higher valuation than expected, and then to compute the valuation recursively using φ_i-adic expansions.

More specifically, if v is a geometric valuation such that $v \succeq v_0$, the concept of a *key polynomial* over v is defined in [Rüt14, Definition 4.7]. Key polynomials are monic polynomials in $\mathcal{O}_K[x]$ — we do not give a definition, which would require more terminology than we need to develop, but see Lemma 4.3 below for the most useful properties. If $\varphi \in \mathcal{O}_K[x]$ is a key polynomial over v, then for $\lambda > v(\varphi)$, we define an *augmented valuation* $v' = [v, v'(\varphi) = \lambda]$ on $K[x]$ by

$$v'(a_0 + a_1\varphi + \cdots + a_r\varphi^r) = \min_{0 \leq i \leq r} v(a_i) + i\lambda$$

whenever the $a_i \in K[x]$ are polynomials with degree less than $\deg(\varphi)$. We should think of this as a "base φ expansion", and of $v'(f)$ as being the minimum valuation of a term in the base φ expansion of f when the valuation of φ is declared to be λ. By [Rüt14, Lemmas 4.11, 4.17], v' is in fact a discrete valuation. In fact, the key polynomials are more or less the polynomials φ for which the construction above yields a discrete valuation for $\lambda > v(\varphi)$. The valuation v' extends to $K(x)$.

We extend this notation to write Mac Lane valuations in the following form:

$$[v_0, v_1(\varphi_1(x)) = \lambda_1, \ldots, v_n(\varphi_n(x)) = \lambda_n].$$

Here each $\varphi_i(x) \in \mathcal{O}_K[x]$ is a key polynomial over v_{i-1}, we have that $\deg(\varphi_{i-1}(x)) \mid \deg(\varphi_i(x))$, and each λ_i satisfies $\lambda_i > v_{i-1}(\varphi_i(x))$. By abuse of notation, we refer to such a valuation as v_n (if we have not given it another name), and we identify v_i with $[v_0, v_1(\varphi_1(x)) = \lambda_1, \ldots, v_i(\varphi_i(x)) = \lambda_i]$ for each $i \leq n$. The valuation v_i is called a *truncation* of v_n. One sees without much difficulty that $v_n(\varphi_i) = \lambda_i$ for all i between 1 and n.

It turns out that set of Mac Lane valuations on $K(x)$ exactly coincides with the set of geometric valuations v with $v \succeq v_0$ ([Rüt14, Theorem 4.31]). Furthermore, every Mac Lane valuation is equal to one where the degrees of the φ_i are strictly increasing ([Rüt14, Remark 4.16]), so we may and do assume this to be the case for the rest of the paper. This has the consequence that the number n is well-defined. We call n the *inductive valuation length* of v. In fact, by [Rüt14, Lemma 4.33], the degrees of the φ_i are invariants of v, once we require that they be strictly increasing. If f is a key polynomial over $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ and either $\deg(f) > \deg(\varphi_n)$ or $v = v_0$, we call f a *proper key polynomial over* v. By our convention, each φ_i is a proper key polynomial over v_{i-1}.

In general, if v and w are two Mac Lane valuations such that the value group Γ_w contains the value group Γ_v, we write $e(w/v)$ for the ramification index $[\Gamma_w : \Gamma_v]$.

Remark 4.1. Observe that if $[v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ is a Mac Lane valuation, where each $\lambda_i = b_i/c_i$ in lowest terms, then the ramification index $e(v_n/v_0)$ equals $\operatorname{lcm}(c_1, \ldots, c_n)$. Consequently, $e(v_i/v_j) = \operatorname{lcm}(c_1, \ldots, c_i)/\operatorname{lcm}(c_1, \ldots, c_j)$ for $i \geq j$.

We collect some basic results on Mac Lane valuations and key polynomials that will be used repeatedly. The following lemma is easy, and we omit its proof.

Lemma 4.2. *Suppose $y = x - a$ with $\nu_K(a) > 0$. If $f = y^d + a_{d-1}y^{d-1} + \cdots + a_0 \in \mathcal{O}_K[y]$ is Eisenstein, then it remains Eisenstein when written as a polynomial in x.*

Lemma 4.3. *Suppose f is a proper key polynomial over $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$.*

- (i) *If $n = 0$, then f is linear. If $n \geq 1$, then φ_1 is linear. Every monic linear polynomial in $\mathcal{O}_K[x]$ is a key polynomial over v_0.*
- (ii) *If $n \geq 1$, and $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ is the φ_n-adic expansion of f, then $v_n(a_0) = v_n(\varphi_n^e) = e\lambda_n$, and $v_n(a_i\varphi_n^i) \geq e\lambda_n$ for all $i \in \{1, \ldots, e-1\}$. In particular, $v_n(f) = e\lambda_n$.*
- (iii) *If $n \geq 1$, then $\deg(f)/\deg(\varphi_n) = e(v_n/v_{n-1})$.*
- (iv) *The product $\deg(f)\lambda_n$ is a positive integer.*
- (v) *If $n \geq 2$, then $\lambda_i > (\deg(\varphi_i)/\deg(\varphi_{i-1}))\lambda_{i-1}$ for all $i \geq 2$. In particular, $\lambda_2 > 1$.*
- (vi) *Suppose the root of φ_1 has positive valuation. The product $\deg(f)\lambda_n$ equals 1 if and only if $n = 1$ and f is Eisenstein.*

Proof. Part (i) follows from [OW18, Remark 5.2(i)] for $n = 0$, and then for general $n \geq 1$ by applying the $n = 0$ case to φ_1 and v_0. Part (ii) follows from [Rüt14, Lemma 4.19(ii), (iii)]. Part (iii) follows from the second equation of [Rüt14, Lemma 4.30], where $\mathbb{F}_m = \mathbb{F}_{m-1} = k$. Note that [Rüt14, Lemma 4.30] is incorrect as stated — the expression $e(v_m/v_{m-1})$ should be replaced by $e(v_{m-1}/v_{m-2})$.

By part (iii) and induction, $\deg(f) = e(v_n/v_0)$. Since $\Gamma_{v_0} = \mathbb{Z}$, it follows that $e(v_n/v_0)v_n(\varphi_n) = \deg(f)\lambda_n$ is an integer, proving (iv).

If $n \geq 2$, then for any $i \geq 2$, we know $\lambda_i > v_{i-1}(\varphi_i)$. Since $\lambda_i = v_i(\varphi_i)$ and since φ_i is a key polynomial over v_{i-1}, part (ii) shows that $v_{i-1}(\varphi_i) = (\deg(\varphi_i)/\deg(\varphi_{i-1}))\lambda_{i-1}$, proving the first part of (v). If $i = 2$, then $\deg(\varphi_{i-1}) = 1$, and the second part of (v) follows from applying part (iv) to φ_2 and λ_1.

If $n > 1$, then $\deg(\varphi_2)\lambda_1$ is an integer by part (iv) applied to v_1 and φ_2. Since $\deg(\varphi_n) > \cdots > \deg(\varphi_2)$, (v) tells us that $\lambda_n > \lambda_{n-1} > \cdots > \lambda_1$. Since $\deg(f) > \deg(\varphi_n)$, putting these together, we get $\deg(f)\lambda_n > \deg(\varphi_2)\lambda_1 \geq 1$ if $n > 1$. Suppose $n = 1$. Then part (i) shows that φ_1 is linear and part (ii) shows that, if $f = \varphi_1^{\deg(f)} + a_{n-1}\varphi_1^{\deg(f)-1} + \cdots + a_1\varphi_1 + a_0$ is the φ_1-adic expansion of f, then $\deg(f)\lambda_1 = v_1(a_0) = \nu_K(a_0)$. Applying Lemma 4.2 with $y = \varphi_1$ proves (vi). ☐

Corollary 4.4. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation of inductive valuation length $n \geq 1$. Write $\lambda_i = b_i/c_i$ in lowest terms for all i. Let $N_n = \operatorname{lcm}_{i<n} c_i$ if $n > 1$, and let $N_n = 1$ if $n = 1$. Then $N_n = e(v_{n-1}/v_0) = \deg(\varphi_n)$, and thus $\Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z} = (1/\deg(\varphi_n))\mathbb{Z}$.*

Proof. That $\deg(\varphi_1) = 1$ is Lemma 4.3(i), which proves the corollary when $n = 1$. By Remark 4.1, $e(v_{j+1}/v_j)\operatorname{lcm}(c_1, \ldots, c_j) = \operatorname{lcm}(c_1, \ldots, c_{j+1})$. The rest of the corollary follows from Lemma 4.3(iii) and induction. ☐

Example 4.5. If $K = \operatorname{Frac}(W(\overline{\mathbb{F}}_3))$, then the polynomial $f(x) = x^3 - 9$ is a proper key polynomial over $[v_0,\, v_1(x) = 2/3]$. In accordance with Lemma 4.3(ii), we have $v_1(f) = v_1(9) = v_1(x^3) = 3 \cdot 2/3 = 2$. If we extend v_1 to a valuation $[v_0,\, v_1(x) = 2/3,\, v_2(f(x)) = \lambda_2]$ with $\lambda_2 > 2$, then the valuation v_2 notices "cancellation" in $x^3 - 9$ that v_1 does not.

4.2. **Mac Lane valuations and diskoids.** Given $\varphi \in \mathcal{O}_K[x]$ monic, irreducible and $\lambda \in \mathbb{Q}_{\geq 0}$, we define the *diskoid* $D(\varphi, \lambda)$ with "center" φ and radius λ to be $D(\varphi, \lambda) := \{\alpha \in \overline{K} \mid \nu_K(\varphi(\alpha)) \geq \lambda\}$ (we only treat diskoids with *non-negative, finite* radius in the sense of [Rüt14, Definition 4.40]). By [Rüt14, Lemma 4.43], a diskoid is a union of a disk with all of its $\operatorname{Gal}(\overline{K}/K)$-conjugates. Such a diskoid is said to be *defined* over K, since $\varphi \in \mathcal{O}_K[x]$. Notice that the *larger* λ is, the *smaller* the diskoid is. We now state the fundamental correspondence between Mac Lane valuations and diskoids.

Proposition 4.6 (cf. [Rüt14, Theorem 4.56])**.** *There is a bijection from the set of diskoids to the set of Mac Lane valuations that sends a diskoid D to the valuation v_D defined by $v_D(f) = \inf_{\alpha \in D} \nu_K(f(\alpha))$. The inverse sends a Mac Lane valuation $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ to the diskoid D_v defined by $D_v = D(\varphi_n, \lambda_n)$. Alternatively,*

$$D_v = \{\alpha \in \overline{K} \;\mid\; \nu_K(f(\alpha)) \geq v(f)\ \forall f \in K[x]\},$$

is a presentation of D_v independent of the description of v as a Mac Lane valuation.

Lastly, if D and D' are diskoids, then $D \subseteq D'$ if and only if $v_D \succeq v_{D'}$. If v and v' are Mac Lane valuations, then $v \succeq v'$ if and only if $D_v \subseteq D_{v'}$.

Lemma 4.7. *Suppose $\alpha \in \overline{K}$, and let f be the minimal K-polynomial for α. For any $\mu \in \mathbb{Q}_{\geq 0}$, let $D(\alpha, \mu)$ be the disk $\{\beta \in \overline{K} \mid \nu_K(\alpha - \beta) \geq \mu\}$. If D is the union of $D(\alpha, \mu)$ with all its $\operatorname{Gal}(\overline{K}/K)$-conjugates, then D is a diskoid of the form $D(f, \lambda)$ for some $\lambda \in \mathbb{Q}_{\geq 0}$.*

Proof. Let $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of f. Reorder the roots $\alpha_2, \ldots, \alpha_n$ so that there exists m with $2 \leq m \leq n$, such that $\nu_K(\alpha_i - \alpha) \geq \mu$ for $2 \leq i \leq m$, and $\nu_K(\alpha_i - \alpha) < \mu$ for $m+1 \leq i \leq n$. Let $\lambda = m\mu + \sum_{i=m+1}^{n} \nu_K(\alpha_i - \alpha)$. We claim that $D = D(f, \lambda)$.

If $\beta \in D(\alpha, \mu)$, then by the non-archimedean triangle inequality, $\nu_K(\alpha_i - \beta) \geq \mu$ for $i \leq m$, and $\nu_K(\alpha_i - \beta) = \nu_K(\alpha_i - \alpha)$ for $i \geq m+1$. Since $\nu_K(f(\beta)) = \sum_{i=1}^{n} \nu_K(\alpha_i - \beta)$, we have $\nu_K(f(\beta)) \geq \lambda$, and thus $\beta \in D(f, \lambda)$. Since $f \in K[x]$, the diskoid $D(f, \lambda)$ is closed under Galois conjugation, so $D \subseteq D(f, \lambda)$.

Now, suppose $\beta \notin D$. By taking a Galois conjugate of β, we may assume that $\nu_K(\alpha - \beta) \geq \nu_K(\alpha_i - \beta)$ for $i \geq 2$. Since $\beta \notin D$, we have that $\nu_K(\alpha - \beta) < \mu$. Thus $\nu_K(\alpha_i - \beta) < \mu$ for $i \leq m$ (indeed, for all i). For $i \geq m+1$, we have $\nu_K(\alpha_i - \alpha) = \nu_K((\alpha_i - \beta) + (\beta - \alpha)) \geq \nu_K(\alpha_i - \beta)$. So $\nu_K(f(\beta)) = \sum_{i=1}^{n} \nu_K(\alpha_i - \beta) < m\mu + \sum_{i=m+1}^{n} \nu_K(\alpha_i - \alpha) = \lambda$. Thus $\beta \notin D(f, \lambda)$, and we are done. ∎

Proposition 4.8. *Let $\alpha \in \mathcal{O}_{\overline{K}}$. If α is contained in a diskoid D defined over K and f is the minimal K-polyomial for α, then there exists $\lambda \in \mathbb{Q}$ such that $D = D(f, \lambda)$.*

Proof. The diskoid D is the union of all Galois conjugates of a disk containing α. By Lemma 4.7, $D = D(f, \lambda)$ for some λ. ∎

The following proposition interprets key polynomials over valuations in terms of diskoids. In particular, a monic irreducible polynomial $f \in \mathcal{O}_K[x]$ is a proper key polynomial over exactly one Mac Lane valuation, whose corresponding diskoid is characterized in part (iii) below.

Proposition 4.9. *Let $\alpha \in \mathcal{O}_{\overline{K}}$, and let $f \in K[x]$ be the minimal polynomial for α.*

(i) *For sufficiently large $\lambda \in \mathbb{Q}$, the diskoid $D(f, \lambda)$ has no elements of lower K-degree than that of α.*

(ii) *There exists a Mac Lane valuation $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ over which f is a proper key polynomial.*

(iii) *If f is a proper key polynomial over a Mac Lane valuation $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ and $n \geq 1$, then the diskoid $D(\varphi_n, \lambda_n)$ corresponding to v is the smallest diskoid containing α and an element of lower K-degree than α.*

(iv) *The Mac Lane valuation from part (ii) is unique.*

Proof. By making λ is sufficiently large, we can force any element of $D(f, \lambda)$ to be arbitrarily close to a K-conjugate of α. Part (i) then follows from Krasner's lemma. Choose such a λ and let $D := D(f, \lambda)$, and let v_D be the corresponding Mac Lane valuation.

To prove (ii), first note that if f is linear, then f is a key polynomial over v_0 by Lemma 4.3(i). So assume $\deg(f) > 1$. Recall the notion of an approximant valuation ([Rüt14, Definition 4.32]): If $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ and w are two Mac Lane valuations, then v is an *approximant* for w if either $v = v_0$, or $v \preceq w$ and $v(g) = w(g)$ for $g = \varphi_n$ and all g such that $\deg(g) < \deg(\varphi_n)$. Let $v = v_n = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ be an approximant to v_D such that $\deg(\varphi_n)$ is as large as possible while being less than $\deg(f)$. We will use the argument on [Mac36, p. 378] (see also [Rüt14, top of p. 29]) to show that f is a proper key polynomial for v; this entails showing two things (a) $v(f) < v_D(f)$ and (b) f has minimal degree amongst polynomials with property (a).

We will now show that $v(f) < v_D(f)$. Now $v \preceq v_D$ implies that $D = D(f, \lambda) \subseteq D_v = D(\varphi_n, \lambda_n)$. Since D does not contain any elements of degree lower than $\deg(f)$ by our choice of λ and since D_v contains the roots of φ_n whereas D does not, $D \subsetneq D_v$. By Proposition 4.8, $D_v = D(f, \lambda')$ for some $\lambda' < \lambda$. Thus $v(f) = \lambda' < \lambda = v_D(f)$.

We then claim that for any ψ such that $v_D(\psi) > v(\psi)$, we have $\deg(\psi) \geq \deg(f)$. Pick such a ψ of lowest degree (which necessarily has to have larger degree than φ_n). By the same argument as in [Rüt14, top of p. 29], ψ will be a key polynomial for v, and the augmented valuation $v' = [v, v'(\psi) = v_D(\psi)]$ is also an approximant to v_D. By our assumption that $\deg(\varphi_n)$ is as large as possible, $\deg(\psi) \geq \deg(f)$.

On the other hand, let $v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$ with $n \geq 1$ be any valuation over which f is a proper key polynomial. Take $\lambda > v(f)$ satisfying the condition of part (i). We may augment v to form the valuation $v_D := [v_0, \ldots, v_n(\varphi_n) = \lambda_n, v_{n+1}(f) = \lambda]$. By Proposition 4.6, we have $D_v = D(\varphi_n, \lambda_n) \supseteq D(f, \lambda) = D$. So D_v contains both α and the roots of φ_n, which have K-degree less than that of α. Suppose $E \subsetneq D_v$ is a smaller diskoid containing α and an element of lower K-degree. Since $\alpha \in E \cap D$ and since D contains no element of lower K-degree than α, [Rüt14, Lemma 4.44] implies then $E \supsetneq D$. By Proposition 4.6, the corresponding comparison $v \prec v_E \prec v_D$ is true on Mac Lane valuations. By the definition of an augmented valuation, $v(g) = v_D(g)$ for all g such that $\deg(g) < \deg(f)$, so $v(g) = v_E(g)$ for such g as well. On the other hand, let $\gamma \in E$ be an element of lower K-degree than α with minimal polynomial h. By the previous sentence, $v(h) = v_E(h)$. Now by Proposition 4.8, D_v and E can respectively be written as $D(h, \mu)$ and $D(h, \mu')$, with $\mu' > \mu$. By Proposition 4.6, $v_E(h) = \mu' > \mu = v(h)$. This is a contradiction, which proves (iii).

If f is linear, then the only Mac Lane valuation over which f is a proper key polynomial is v_0 by Lemma 4.3(i). If $\deg(f) > 1$ and f is a proper key polynomial over v as in part (ii), then by part (iii), D_v can be characterized as the smallest diskoid containing α and an element of lower K-degree. So D_v (and thus v) are characterized completely in terms of f. This proves (iv). □

To close out §4.2, we prove several results linking Mac Lane valuations evaluated at a polynomial to the valuation of that polynomial at a particular point.

Definition 4.10 ([Rüt14, Definition 4.4, Lemma 4.24])**.** If $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ is a Mac Lane valuation and $f \in K[x]$, then a *v-reciprocal* of f is a polynomial $f' \in K[x]$ such that $v(ff' - 1) > 0$ and $v(f') = v_{n-1}(f') = -v(f)$.

By [Rüt14, Lemma 4.24], any $f \in K[x]$ with $v(f) = v_{n-1}(f)$ has a v-reciprocal. In this case, it is clear from Definition 4.10 that f and f' being v-reciprocals is a symmetric relation.

Proposition 4.11. *Suppose $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ is a Mac Lane valuation, $\alpha \in D(\varphi_n, \lambda_n)$, and $g \in K[x]$ such that $v(g) = v_{n-1}(g)$. Then $\nu_K(g(\alpha)) = v(g)$.*

Proof. Let $D := D(\varphi_n, \lambda_n)$ be the diskoid corresponding to v and let $D' := D(g, \nu_K(g(\alpha)))$ with corresponding valuation v'. These two diskoids share the common element α. By [Rüt14, Lemma 4.44], either $D \subseteq D'$ or $D' \subseteq D$, and then Proposition 4.6 shows that either $v' \preceq v$ or $v \preceq v'$.

Suppose $\nu_K(g(\alpha)) > v(g)$. Since $\alpha \in D$, by Proposition 4.6 we have $\nu_K(g(\alpha)) \geq v(g)$. Since $v'(g) = \nu_K(g(\alpha))$ by definition, we have $v(g) \lneq v'(g)$. Since either $v' \preceq v$ or $v \preceq v'$, it follows that $v \preceq v'$. Let $g' \in K[x]$ be a v-reciprocal of g, i.e., $gg' = 1 + h$ with $v(h) > 0$ (g' exists because $v(g) = v_{n-1}(g)$). Since $v \preceq v'$, we have $0 < v(h) \leq v'(h)$. In particular, $v'(gg') = v(gg') = 0$, so $v'(g') = -v'(g) < -v(g) = v(g')$. But this contradicts $v \preceq v'$. □

Corollary 4.12. *If f is a key polynomial over $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \dots,\, v_n(\varphi_n) = \lambda_n]$ with root $\alpha \in \overline{K}$, then $\nu_K(g(\alpha)) = v(g)$ for all $g \in \mathcal{O}_K[x]$ of degree less than* $\deg(f)$. *In particular, $\nu_K(\varphi_i(\alpha)) = \lambda_i$ for all $1 \le i \le n$.*

Proof. Consider a Mac Lane valuation $w_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \dots,\, v_n(\varphi_n) = \lambda_n,\, v_{n+1}(f) = \lambda_{n+1}]$, with λ_{n+1} large. Then $v_{n+1}(g) = v_n(g)$ and $\alpha \in D(f, \lambda_{n+1})$, so the corollary follows from Proposition 4.11. ☐

The following corollary is only used to prove Lemma 6.5.

Corollary 4.13. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \dots,\, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation. If $a_e\varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ is the φ_n-adic expansion of a polynomial $f \in K[x]$ and if $v(f) = v(a_0)$, then f has no roots α such that $\nu_K(\varphi_n(\alpha)) > \lambda_n$.*

Proof. Observe that since $v(f) = v(a_0)$, we have $v(a_0) \le v(a_i\varphi_n^i)$ for all $i > 0$. Also, by the definition of v, we have $v(a_i) = v_{n-1}(a_i)$ for all $0 \le i \le e$. If α is a root of f such that $\nu_K(\varphi_n(\alpha)) > \lambda_n = v(\varphi_n)$, then $\alpha \in D(\varphi_n, \lambda_n)$. Combining the last two sentences with Proposition 4.11 gives $\nu_K(a_i(\alpha)) = v(a_i)$. So $\nu_K(a_i(\alpha)\varphi_n(\alpha)^i) > v(a_i\varphi_n^i) \ge v(a_0) = v(a_0(\alpha))$ for $i > 0$. Since $a_0(\alpha)$ is the unique term of the φ_n-adic expansion of $f(\alpha)$ with lowest valuation, α cannot be a root of f. ☐

5. Mac Lane valuations, normal models and regular resolutions

In this section (§5.2 and §5.3), we will explicitly describe certain regular models of $\mathbb{P}^1_K$ using Mac Lane valuations, and count the number of components in their special fiber. In §5.4, we discuss regular models associated to a particular polynomial f. These models will then be used in §6 in the construction of models of $\mathbb{P}^1_K$ that help realize the conductor-discriminant inequality for f. Before this, §5.1 contains results on the specialization of horizontal divisors, expressed in terms of Mac Lane valuations, which will be used throughout the rest of the paper.

A *normal model* of $\mathbb{P}^1_K$ is a flat, normal, proper $\mathcal{O}_K$-curve with generic fiber isomorphic to $\mathbb{P}^1_K$. By [Rüt14, Proposition 3.4], normal models $\mathcal{Y}$ of $\mathbb{P}^1_K$ are in one-to-one correspondence with non-empty finite collections of geometric valuations, by sending $\mathcal{Y}$ to the collection of geometric valuations corresponding to the local rings at the generic points of the irreducible components of the special fiber of $\mathcal{Y}$. Via this correspondence, the multiplicity of an irreducible component of the special fiber of a normal model $\mathcal{Y}$ of $\mathbb{P}^1_K$ corresponding to a Mac Lane valuation v equals $e(v/v_0)$.

We say that a normal model of $\mathbb{P}^1_K$ *includes* a Mac Lane valuation v if a component of the special fiber corresponds to v. If $\mathcal{Y}$ includes v, we call the corresponding irreducible component of its special fiber the *v-component* of the special fiber of $\mathcal{Y}$ (or simply the v-component of $\mathcal{Y}$, even though it is not an irreducible component of $\mathcal{Y}$). If S is a finite set of Mac Lane valuations, then the *S-model of* $\mathbb{P}^1_K$ is the normal model including exactly the valuations in S. If $S = \{v\}$, we simply say the v-model instead of the $\{v\}$-model. Recall that we fixed a coordinate x on $\mathbb{P}^1_K$, that is, a rational function x on $\mathbb{P}^1_K$ such that $K(\mathbb{P}^1_K) = K(x)$.

5.1. Specialization of horizontal divisors.

Each $\alpha \in \overline{K} \cup \{\infty\}$ corresponds to a point of $\mathbb{P}^1(\overline{K})$ given by $x = \alpha$, which lies over a unique closed point of $\mathbb{P}^1_K$. If $\mathcal{Y}$ is a normal model of $\mathbb{P}^1_K$, the closure of this point in $\mathcal{Y}$ is a subscheme that we call D_α; note that D_α

is a horizontal divisor (the model will be clear from context, so we omit it to lighten the notation).

If v is a Mac Lane valuation, then the reduced special fiber of the v-model of $\mathbb{P}^1_K$ is isomorphic to $\mathbb{P}^1_k$ (see, e.g., [OW18, Lemma 7.1]). It will occasionally be useful to have an explicit coordinate on this special fiber (that is, a rational function y such that the function field of the special fiber is $k(y)$).

Lemma 5.1. *Let $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation, and let $e = e(v_n/v_{n-1})$. There exists a monomial t in $\varphi_1, \ldots, \varphi_{n-1}$ such that $v(t\varphi_n^e) = 0$, and for any such t, the restriction of $t\varphi_n^e$ to the reduced special fiber of the v-model of $\mathbb{P}^1_K$ is a coordinate on the v-component that vanishes at the specialization of $\varphi_n = 0$.*

Proof. Let $\mathcal{O} \subseteq K[x]$ be the subring of elements f such that $v(f) \geq 0$, and let $\mathcal{O}^+$ be the ideal of elements g where $v(g) > 0$. Let $e = e(v_n/v_{n-1})$. By [Rüt14, Lemma 4.29] and the discussion before that lemma, $\mathcal{O}/\mathcal{O}^+ \cong k[y]$, where y is the image of $t\varphi_n^e$ in $\mathcal{O}/\mathcal{O}^+$, for any $t \in K[x]$ with $v(t\varphi_n^e) = 0$ and $v(t) = v_{n-1}(t)$ (in the notation of [Rüt14], the example used is $t = (S')^\ell$). Since $v(\varphi_n^e) \in \Gamma_{v_{n-1}}$, we can take t to be a monomial in $\varphi_1, \ldots, \varphi_{n-1}$. Since Spec $\mathcal{O}$ is an affine open of the v-model with reduced special fiber Spec $\mathcal{O}/\mathcal{O}^+ \cong$ Spec $k[y] \cong \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$, we have that y is a coordinate on the reduced special fiber of the v-model of $\mathbb{P}^1_K$. □

Proposition 5.2. *Let $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation and let $\mathcal{Y}$ be the v-model of $\mathbb{P}^1_K$. As α ranges over $\overline{K}$, all D_α with $\nu_K(\varphi_n(\alpha)) > \lambda_n$ meet on the special fiber, all D_α with $\nu_K(\varphi_n(\alpha)) < \lambda_n$ meet at a different point on the special fiber, and no D_α with $\nu_K(\varphi_n(\alpha)) \neq \lambda_n$ meets any D_β with $\nu_K(\varphi_n(\beta)) = \lambda_n$.*

Proof. Let $\mathcal{Y}$ be the v-model of $\mathbb{P}^1_K$. Using the coordinate $y := t\varphi_n^e$ from Lemma 5.1 on the reduced special fiber of $\mathcal{Y}$, we will show that all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) < \lambda_n$ specialize to $y = \infty$, all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) > \lambda_n$ specialize to $y = 0$ and all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) = \lambda_n$ specialize to some point $y = a$ with $a \notin \{0, \infty\}$. We now work out the details.

Let $\mathcal{O} \subseteq K[x]$ be the subring of elements f such that $v(f) \geq 0$, and let $\mathcal{O}^+$ be the ideal of elements g where $v(g) > 0$. Suppose $\alpha \in D(\varphi_n, \lambda_n)$. Proposition 4.6 shows that $\nu_K(g(\alpha)) > 0$ for $g \in \mathcal{O}^+$, thus evaluating y at α gives a well-defined element of k. Furthermore, $y = y(\alpha)$ is precisely the point where D_α meets the special fiber of $\mathcal{Y}$. We now compute:

$$\begin{aligned} y(\alpha) = 0 &\Leftrightarrow \nu_K(t(\alpha)\varphi_n(\alpha)^e) > 0 \\ &\Leftrightarrow \nu_K(t(\alpha)\varphi_n(\alpha)^e) > v(t\varphi_n^e) \\ &\Leftrightarrow \nu_K(\varphi_n(\alpha)) > \lambda_n \qquad\qquad (\because\ \nu_K(t(\alpha)) = v(t)). \end{aligned}$$

This shows that all D_α for which $\nu_K(\varphi_n(\alpha)) > \lambda_n$ intersect on the special fiber at the point $y = 0$, but none of them intersect any D_β for which $\nu_K(\varphi_n(\beta)) = \lambda_n$. All such D_β intersect the reduced special fiber $\mathbb{A}^1_k \cong$ Spec $k[y]$ of Spec $\mathcal{O}$ at some point where $y \neq 0$.

Now let $\alpha \notin D(\varphi_n, \lambda_n)$. We will show that $D_\alpha \cap (\text{Spec } \mathcal{O})_s$ is empty by contradiction. Suppose not. Let $P \in D_\alpha \cap (\text{Spec } \mathcal{O})_s$ be a closed point of Spec $\mathcal{O}$. We have a well-defined element $g(P) \in k$ for every $g \in \mathcal{O}$ coming from evaluating g at P. Since P is the closed point of $D_\alpha \cong$ Spec A with $A \subseteq \mathcal{O}_{K(\alpha)}$, it follows that $g(\alpha) \in \mathcal{O}_{K(\alpha)}$ and furthermore, $g(P) = g(\alpha)$ mod $\mathfrak{m}_{\mathcal{O}_{K(\alpha)}}$. We will now construct a $g \in \mathcal{O}$ with $\nu_K(g(\alpha)) < 0$, which is a contradiction.

Let b be such that $bv(\varphi_n) \in \mathbb{Z}_{>0}$, and let $g := \varphi_n^b / \pi_K^{bv(\varphi_n)}$. Then $v(g) = 0$ so $g \in \mathcal{O}$, but

$$\nu_K(g(\alpha)) = b(\nu_K(\varphi_n(\alpha)) - v(\varphi_n)) < 0.$$

Thus D_α does not intersect the special fiber of Spec $\mathcal{O}$, so D_α specializes to a point of $\mathcal{Y}_s \setminus (\text{Spec } \mathcal{O})_s$, which is the "point at infinity" where $y = \infty$ on the reduced special fiber of $\mathcal{Y}$. This finishes the proof. □

Corollary 5.3. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation and let $\mathcal{Y}$ be a normal model of $\mathbb{P}^1_K$ including v. If $\alpha, \beta \in \overline{K}$ are such that $\nu_K(\varphi_n(\beta)) \leq \lambda_n \leq \nu_K(\varphi_n(\alpha))$ and $\nu_K(\varphi_n(\beta)) \neq \nu_K(\varphi_n(\alpha))$, then D_α and D_β do not meet on the special fiber of $\mathcal{Y}$.*

Proof. Immediate from Proposition 5.2. □

Corollary 5.4. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ and $v' = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda'_n]$ be Mac Lane valuations with $\lambda'_n < \lambda_n$, and let $\mathcal{Y}$ be the $\{v, v'\}$-model of $\mathbb{P}^1_K$. Then D_α meets the intersection point of the two irreducible components of the special fiber of $\mathcal{Y}$ if and only if $\lambda'_n < \nu_K(\varphi_n(\alpha)) < \lambda_n$.*

Proof. Let $\overline{Y}$ and $\overline{Y}'$ be the v and v'-components of $\mathcal{Y}$, respectively, and let $z := \overline{Y} \cap \overline{Y}'$. First suppose $\lambda'_n < \nu_K(\varphi_n(\alpha)) < \lambda_n$. If D_α meets a point of $\overline{Y} \setminus \overline{Y}'$, then by Proposition 5.2 applied to the blow down of $\overline{Y}' \subseteq \mathcal{Y}$ (i.e., the v-model of $\mathbb{P}^1_K$), all D_α outside of $D(\varphi_n, \lambda_n)$ intersect this point on $\overline{Y} \subseteq \mathcal{Y}$. So if we blow down $\overline{Y} \subseteq \mathcal{Y}$, then all D_α for $\alpha \notin D(\varphi_n, \lambda_n)$ specialize to the same point. Since we can find $\alpha_1, \alpha_2 \in \overline{K} \setminus D(\varphi_n, \lambda_n)$ with $\nu_K(\varphi_n(\alpha_1)) = \lambda'_n$ and $\lambda'_n < \nu_K(\varphi_n(\alpha_2)) < \lambda_n$, the previous line contradicts Proposition 5.2 applied to the v'-model of $\mathbb{P}^1_K$. The same argument applied to the blow down of $\overline{Y}$ (i.e, the v-model of $\mathbb{P}^1_K$) yields a contradiction if D_α intersects a point of $\overline{Y}' \setminus \overline{Y}$. So D_α meets the intersection point z of the two irreducible components of the special fiber.

Fix $\beta \in \overline{K}$ such that $\lambda'_n < \nu_K(\varphi_n(\beta)) < \lambda_n$. If $\nu_K(\varphi_n(\alpha)) \leq \lambda'_n$, then Corollary 5.3 shows that D_α and D_β do not meet on the v'-model of $\mathbb{P}^1_K$, and thus not on $\mathcal{Y}$ either. In particular, since D_β meets z by the previous paragraph, D_α does not. A similar proof works if $\nu_K(\varphi_n(\alpha)) \geq \lambda_n$ using the v-model instead of the v'-model. □

5.2. **Resolution of singularities on normal models of $\mathbb{P}^1$.** Let $\mathcal{Y}$ be a normal model of $\mathbb{P}^1_K$. A *minimal regular resolution of* $\mathcal{Y}$ is a (proper) regular model $\mathcal{Z}$ of $\mathbb{P}^1_K$ with a surjective, birational morphism $\pi : \mathcal{Z} \to \mathcal{Y}$ such that the special fiber of $\mathcal{Z}$ contains no -1-components ([CES03, Definition 2.2.1]). Such minimal regular resolutions exist and are unique, e.g., by [CES03, Theorem 2.2.2]. We record the following lemma, that we will need in §9 and §10.

Lemma 5.5. *Suppose S and T are finite sets of Mac Lane valuations with $S \cap T \neq \emptyset$. If both the S-model and the T-model of $\mathbb{P}^1_K$ are regular, then the $(S \cup T)$-model of $\mathbb{P}^1_K$ is also regular.*

Proof. Let $U = S \cap T$ and $V = S \cup T$. Let $\mathcal{Y}_S$, $\mathcal{Y}_T$, $\mathcal{Y}_U$, and $\mathcal{Y}_V$ be the S-, T-, U-, and V-models of $\mathbb{P}^1_K$. We first claim that $\mathcal{Y}_U$ is regular. Its minimal regular resolution corresponds to a finite set W of Mac Lane valuations. Since $\mathcal{Y}_S$ and $\mathcal{Y}_T$ are regular, $U \subseteq W \subseteq S$ and $U \subseteq W \subseteq T$. So $W = U$ and the claim is proved.

Now, since $\mathcal{Y}_S, \mathcal{Y}_T, \mathcal{Y}_U$ are all regular, it follows that $p_S : \mathcal{Y}_S \to \mathcal{Y}_U$ and $p_T : \mathcal{Y}_T \to \mathcal{Y}_U$ each come from successive blowups at closed points. We also have that the Mac Lane valuations corresponding to the exceptional divisors E_S of p_S and E_T of p_T are disjoint by assumption. This means that $p_S(E_S)$ and $p_T(E_T)$ are disjoint on $\mathcal{Y}_U$ for otherwise, one could start the process of blowing up $\mathcal{Y}_U$ at the same point to get either $\mathcal{Y}_S$ or $\mathcal{Y}_T$, which would result in a common Mac Lane valuation. Thus $\mathcal{Y}_V = \mathcal{Y}_S \times_{\mathcal{Y}_U} \mathcal{Y}_T$. Since every point on $\mathcal{Y}_V$ has a local neighborhood isomorphic to that of a point on $\mathcal{Y}_S$ or $\mathcal{Y}_T$, we conclude that $\mathcal{Y}_V$ is regular. □

In the remainder of §5.2, we recall a fundamental result from [OW18], which expresses minimal regular resolutions of models of $\mathbb{P}^1_K$ with irreducible special fiber in terms of Mac Lane valuations. We then give upper bounds on numbers of irreducible components on the special fiber of such a resolution.

5.2.1. *Shortest N-paths.* We start by recalling the notion of a *shortest N-path*, introduced in [OW18].

Definition 5.6. Let N be a natural number, and let $a > a' \geq 0$ be rational numbers. An *N-path* from a to a' is a decreasing sequence $a = b_0/c_0 > b_1/c_1 > \cdots > b_r/c_r = a'$ of rational numbers in lowest terms such that

$$\frac{b_i}{c_i} - \frac{b_{i+1}}{c_{i+1}} = \frac{N}{\operatorname{lcm}(N, c_i)\operatorname{lcm}(N, c_{i+1})}$$

for $0 \leq i \leq r-1$. If, in addition, no proper subsequence of $b_0/c_0 > \cdots > b_r/c_r$ containing b_0/c_0 and b_r/c_r is an N-path, then the sequence is called the *shortest N-path* from a to a'.

Remark 5.7. By [OW18, Proposition A.14], the shortest N-path from a' to a exists and is unique.

Remark 5.8. Observe that two successive entries $b_i/c_i > b_{i+1}/c_{i+1}$ of a shortest 1 path satisfy $b_i/c_i - b_{i+1}/c_{i+1} = 1/(c_i c_{i+1})$.

The construction of shortest N-paths in [OW18] proceeded using negative continued fractions. We give an equivalent construction here for certain 1-paths.

Proposition 5.9. *Let $b/c \in [0,1]$ be a fraction written in lowest terms. Let $1/1 = b_r/c_r > b_{r-1}/c_{r-1} > \cdots > b_0/c_0 = b/c$ be the shortest 1-path from 1 to b/c with all entries written in lowest terms. Let $b/c = b'_0/c'_0 > b'_1/c'_1 > \cdots > b'_{r'}/c'_{r'} = 0/1$ be the shortest 1-path from b/c to 0 with all entries written in lowest terms.*

(i) *For $0 \leq i \leq r-1$, the fraction b_{i+1}/c_{i+1} is the entry following b_i/c_i in the Farey sequence with denominator bounded by c_i. In particular, $1 = c_r < c_{r-1} < \cdots < c_0 = c$.*
(ii) *For $0 \leq i \leq r'-1$, the fraction b'_{i+1}/c'_{i+1} is the entry preceeding b'_i/c'_i in the Farey sequence with denominator bounded by c'_i. In particular, $c = c'_0 > c'_1 > \cdots > c'_{r'} = 1$.*
(iii) *Assume that $0 < b/c < 1$. Then there exists a least index i such that $c_i < c'_1$. Furthermore, we have that b_j/c_j is the mediant of b_{j+1}/c_{j+1} and b'_1/c'_1 for all $j \in \{0, \ldots, i-1\}$. In particular, b/c is the mediant of b_1/c_1 and b'_1/c'_1, and $c_0, c_1, \ldots, c_i$ form an arithmetic progression with increment $-c'_1$.*

Proof. We first prove (ii). Let P be a path from b/c to 0 constructed by taking neighboring entries in Farey sequences as in the proposition. It suffices to prove that P is a shortest 1-path. It is well known that neighboring entries in Farey sequences satisfy the criterion to be part of a 1-path. Now, suppose we have a segment of P given by $e/f > g/h > i/j$ in lowest terms. By construction of P, it is clear that $f > h > j$. Note that $e/f - i/j =$

$(e/f - g/h) + (g/h - i/j) = 1/fh + 1/hj = (f + j)/fhj$. Since $(f + j)/h > 1$, we have $e/f - i/j \neq 1/fj$. So we cannot remove g/h while keeping P a 1-path. Thus P is a shortest 1-path. This proves (ii). The proof of (i) is exactly the same, except that the successive denominators are increasing instead of decreasing.

Now assume that $0 < b/c < 1$. Since $b/c \neq 0$, by Remark 5.8, it follows that $c'_1 \neq 1$. Since $1 = c_r < c_{r-1} < \cdots < c_0 = c$ and $c = c'_0 > c'_1 > \cdots > c'_{r'} = 1$ by parts (i) and (ii), it follows that the least index i with $c_i < c'_1$ is well-defined. By (i) and (ii), b/c lies directly between b'_1/c'_1 and b_1/c_1 in the Farey sequence with denominator bounded by c. In general, for $j \in \{0, \ldots, i-1\}$, we have $c_0 > c_1 > \cdots > c_j \geq c'_1$, so it is also true that b'_1/c'_1 is entry preceeding b_j/c_j in the Farey sequence with denominator bounded by c_j. Applying part (i) to b_j/c_j, we have that b_{j+1}/c_{j+1} is the entry following b_j/c_j in this same Farey sequence. By well-known properties of the Farey sequence, b_j/c_j is the mediant of b_{j+1}/c_{j+1} and b'_1/c'_1. The statement about arithmetic progressions follows immediately. This proves (iii). □

Example 5.10. The sequence $1 > 1/2 > 2/5 > 3/8 > 1/3 > 0$ is a concatenation of the shortest 1-path from 1 to 3/8 with the shortest 1-path from 3/8 to 0. Note that the denominators increase until 3/8 and then decrease afterwards. As Proposition 5.9(iii) guarantees, 3/8 is the mediant of 2/5 and 1/3, and 2/5 is the mediant of 1/2 and 1/3. The denominators 8, 5, and 2 form an arithmetic progression with increment -3.

Corollary 5.11.

(i) *The shortest* 1*-path from* 1 *to* b/c *contains at most* $c - b + 1$ *entries. The shortest* 1*-path from* b/c *to* 0 *contains at most* $b + 1$ *entries.*

(ii) *If* $c \equiv 1 \pmod 3$ *and* $c \geq 7$*, then the shortest* 1*-path from* 1 *to* $3/c$ *contains at most* $c - 3$ *entries. If* $c \equiv 2 \pmod 3$*, then the shortest* 1*-path from* $3/c$ *to* 0 *contains* 3 *entries.*

Proof. To prove the first statement of (i), it suffices to show that if $b''/c'' > b'/c'$ are two successive entries of the shortest 1-path from 1 to b/c, then $c' - b' > c'' - b''$. Since $1 > b''/c'' > b'/c'$, it follows that $(c' - b')/c' > (c'' - b'')/c'' > 0$, which in turn implies that $(c' - b') > (c'' - b'') \cdot c'/c'' > 0$. Since b''/c'' follows b'/c' in a Farey sequence with denominators bounded by c' (Proposition 5.9(i)), we have $c' > c''$. Putting the last two inequalities together, we get $c' - b' > c'' - b''$, which proves the first statement of (i).

To prove the second statement of (i), it suffices to show that if $b''/c'' > b'/c'$ are two successive entries in lowest terms of the shortest 1-path from b/c to 0, then $b' < b''$. Again, since b'/c' preceeds b''/c'' in a Farey sequence (Proposition 5.9(ii)), $c' < c''$. Since $b'/c' < b''/c''$, we conclude that $b' < b''$.

To prove the first statement of (ii), note that if P is the shortest 1-path from 1 to $1/((c-1)/3)$, then concatenating $3/c$ to the end of P gives a 1-path from 1 to $3/c$. By part (i), the shortest 1-path from 1 to $1/((c-1)/3))$ contains at most $(c-1)/3$ elements. So the shortest 1-path from 1 to $3/c$ contains at most $(c-1)/3 + 1$ elements, which is at most $c - 3$ when $c \geq 7$. To prove the second statement, we explicitly write the shortest 1-path from $3/c$ to 0: It is $3/c > 1/((c+1)/3) > 0$. □

5.2.2. *Regular resolutions.* The following proposition expresses minimal regular resolutions in terms of Mac Lane valuations and shortest N-paths. It is fundamental to our calculation.

Proposition 5.12 ([OW18, Theorem 7.8])**.** *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$. For each i, write $\lambda_i = b_i/c_i$ in lowest terms, and let $N_i = \operatorname{lcm}_{j<i} c_j = \deg(\varphi_i)$ (Corollary 4.4). Set $\lambda_0 = \lfloor \lambda_1 \rfloor$, as well as $N_0 = N_1 = 1$ and $e(v_0/v_{-1}) = 1$. Let $\mathcal{Y}_v$ be the v-model of $\mathbb{P}^1_K$. Then the minimal regular resolution of $\mathcal{Y}_v$ is the normal model of $\mathbb{P}^1_K$ that includes exactly the following set of valuations:*

- *For each $1 \leq i \leq n$, the valuations*
$$v_{i,\lambda} := [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{i-1}(\varphi_{i-1}) = \lambda_{i-1},\, v_i(\varphi_i) = \lambda],$$
as λ ranges through the shortest N_i-path from β_i to λ_i, where β_i is the least rational number greater than or equal to λ_i in $(1/N_i)\mathbb{Z} = \Gamma_{v_{i-1}}$.
- *For each $0 \leq i \leq n-1$, the valuations*
$$w_{i,\lambda} := [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_i(\varphi_i) = \lambda_i,\; v_{i+1}(\varphi_{i+1}) = \lambda],$$
as λ ranges through the shortest N_{i+1}-path from λ_{i+1} to $e(v_i/v_{i-1})\lambda_i$, excluding the endpoints.
- *The valuation $\tilde{v}_0 := [v_0,\, v_1(\varphi_1) = \lambda_0]$ (which is just v_0 if $\lambda_1 < 1$).*

Remark 5.13. If v_i is a truncation of the Mac Lane valuation v, then $v_i = v_{i,\lambda_i}$ for $i \geq 1$. In particular, the minimal regular resolution of $\mathcal{Y}_v$ includes v_i for $i \geq 1$.

Remark 5.14. Since $v_i(\varphi_{i+1}) = e(v_i/v_{i-1})\lambda_i$ by parts (ii) and (iii) of Lemma 4.3 applied to φ_i and $\lambda_{i+1} > v_i(\varphi_{i+1})$ by the definition of a Mac Lane valuation, we have $\lambda_{i+1} > e(v_i/v_{i-1})\lambda_i$. This it makes sense to talk about the N_{i+1}-path from λ_{i+1} to $e(v_i/v_{i-1})\lambda_i$.

Remark 5.15. For v as in Proposition 5.12, consider the set S of valuations included in the minimal regular resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of the v-model $\mathcal{Y}_v$ of $\mathbb{P}^1_K$. Using the partial order $\prec$ on S, one constructs a tree whose vertices are the elements of S and where there is an edge between two vertices w and w' if and only if $w \prec w'$ and there is no w'' with $w \prec w'' \prec w'$. One can show that this tree is the dual graph of $\mathcal{Y}_v^{\mathrm{reg}}$. This graph is shown in Figure 3.

Corollary 5.16. *Let $\mathcal{Y}_v^{\mathrm{reg}}$ be the minimal resolution of the v-model of $\mathbb{P}^1_K$ as in Proposition 5.12, where $v \neq v_0$. Let $\mathcal{Y}_0$ be the minimal regular resolution of the $\{v, v_0\}$-model of $\mathbb{P}^1_K$. Assume that the roots of φ_n have positive valuation.*

(i) *The valuations included in $\mathcal{Y}_0$ are the valuations included in $\mathcal{Y}_v^{\mathrm{reg}}$ as well as v_0 and the valuations $[v_0,\, v_1(\varphi_1) = \lambda]$ for $\lambda \in \{1, 2, \ldots, \lambda_0 - 1\}$. Equivalently, the valuations included in $\mathcal{Y}_0$ are exactly the valuations we would get from Proposition 5.12 if we changed our convention from $\lambda_0 = \lfloor \lambda_1 \rfloor$ to $\lambda_0 = 0$.*
(ii) *We have $\mathcal{Y}_v^{\mathrm{reg}} = \mathcal{Y}_0$ if and only if $\lambda_1 < 1$.*
(iii) *The model $\mathcal{Y}_0$ includes the valuation $[v_0,\, v_1(x) = 1/d]$ for some $d \in \mathbb{N}$.*
(iv) *For d as in part (iii), $\mathcal{Y}_0$ also includes $[v_0,\, v_1(x) = 1/c]$ for all $1 \leq c < d$. In particular, $\mathcal{Y}_0$ includes $[v_0,\, v_1(x) = 1]$.*

Proof. If $\lambda_1 < 1$, then part (i) is vacuous. If $\lambda_0 \geq 1$, then if $\mathcal{Z}$ is the normal model of $\mathbb{P}^1_K$ including the valuations included in $\mathcal{Y}$ as well as v_0, then there may be a singularity where the components corresponding to v_0 and $[v_0,\, v_1(\varphi_1) = \lambda_0]$ cross. Since v_0 and $[v_0,\, v_1(\varphi_1) = 0]$ are the same valuation, and since $\lambda_0 > \lambda_0 - 1 > \cdots > 1 > 0$ is the shortest 1-path from λ_0 to 0, [OW18, Corollary 7.5] shows that resolving this singularity yields exactly the description of $\mathcal{Y}_0$ in part (i). Part (ii) is clear, since $\lambda_1 < 1$ is equivalent to $\lambda_0 = 0$.

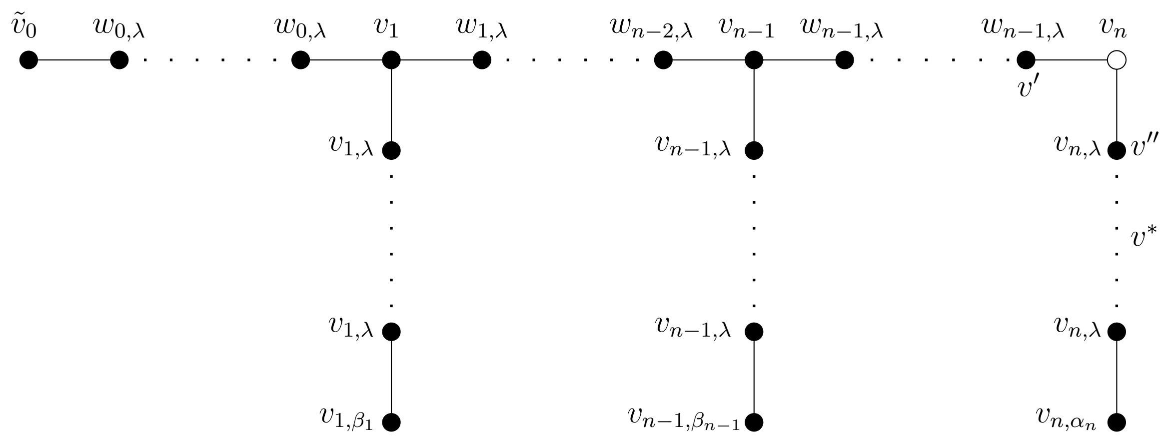


FIGURE 3. The dual graph of the minimal resolution of the $v = v_n$-model of $\mathbb{P}^1_K$. The white vertex corresponds to the strict transform of the v. The vertices labeled v' and v'' correspond to the successor and precursor valuations of v_n, see §5.3. The valuation v^* in that section corresponds to some black vertex in the rightmost column, possibly equal to v''.

If $\lambda_0 \geq 1$, we have already seen that $\mathcal{Y}_0$ includes $[v_0,\, v_1(\varphi_1(x)) = 1]$. Now, the values of λ corresponding to the valuations $w_{0,\lambda}$ from Proposition 5.12 form the shortest 1-path from λ_1 to λ_0. If $\lambda_0 = 0$, then the definition of a 1-path shows that the entry preceeding λ_0 in this 1-path is of the form $1/d$ for some $d \in \mathbb{N}$. So $\mathcal{Y}_0$ includes $[v_0,\, v_1(\varphi_1(x)) = 1/d]$.

Since the roots of φ_n have positive valuation, and $D(\varphi_n, \lambda_n) \subseteq D(\varphi_1, \lambda_1)$, it follows that the diskoid $D(\varphi_1, \lambda_1)$ contains elements with positive valuation. Since $\lambda_1 > 0$ and $\varphi_1 = x - \alpha$ for some $\alpha \in \mathcal{O}_K$, it follows that $D(\varphi_1, \lambda_1)$ can contain an element of positive valuation only if $\nu_K(\alpha) > 0$. This means that $\nu_K(\alpha) \geq 1$. By [Rüt14, Lemma 4.33(c)], $[v_0,\, v_1(\varphi_1(x)) = 1/c] = [v_0,\, v_1(x) = 1/c]$ for all $c \in \mathbb{N}$. Combining this with the previous paragraph proves part (iii).

Since $\mathcal{Y}_0$ is regular, if it includes $[v_0,\, v_1(x) = 1/d]$, then it includes all valuations corresponding to components on the special fiber of its minimal regular resolution. By Proposition 5.12, this includes the $v_{1,\lambda}$, which are valuations of the form $[v_0,\, v_1(x) = \lambda]$, as λ ranges over the shortest 1-path from 1 to $1/d$. One checks that this path is $1 > 1/2 > 1/3 > \cdots > 1/d$, proving part (iv). □

Corollary 5.17. *Let v and $\mathcal{Y}_v$ be as in Proposition 5.12, and use the notation there.*

(i) *For $1 \leq i \leq n$, the number of valuations $v_{i,\lambda}$ included in the minimal resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of $\mathcal{Y}_v$ is bounded above by $e(v_i/v_{i-1})$.*

(ii) *For $1 \leq i \leq n$, the total number of valuations among the $w_{i-1,\lambda}$ and the $v_{i,\lambda}$ included in the minimal resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of $\mathcal{Y}_v$ is bounded above by $\lfloor N_i\lambda_i \rfloor - N_i(e(v_{i-1}/v_{i-2})\lambda_{i-1}) + e(v_i/v_{i-1})$. If $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, then the total number of valuations for $i = 1$ above is in fact at most $e(v_1/v_0) - 1$ (as opposed to $e(v_1/v_0)$).*

(iii) *If $\lambda_n \in (1/N_n)\mathbb{Z} = \Gamma_{v_{n-1}}$ (Corollary 4.4), then the total number of valuations among the $w_{n-1,\lambda}$ and the $v_{n,\lambda}$ included in the minimal resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of $\mathcal{Y}_v$ is bounded above by $N_n(\lambda_n - e(v_{n-1}/v_{n-2})\lambda_{n-1})$.*

(iv) *Parts (i), (ii), and (iii) hold with* $\mathcal{Y}_0$ *from Corollary 5.16 replacing* $\mathcal{Y}_v^{\mathrm{reg}}$*, provided that we replace* λ_0 *by* 0 *in parts (ii) and (iii) above.*

Proof. By Proposition 5.12, to prove part (i), we bound the length of the shortest N_i-path from β_i to λ_i. If $N_i\lambda_i \in \mathbb{Z}$, then this path has length 1 and part (i) holds. Now assume that $N_i\lambda_i \notin \mathbb{Z}$. By [OW18, Lemma A.7], multiplying all the entries by N_i gives the shortest 1-path from $N_i\beta_i$ to $N_i\lambda_i$. By the definition of β_i, we have that $N_i\beta_i = \lceil N_i\lambda_i \rceil$, and the denominator of $N_i\lambda_i$ is $\mathrm{lcm}(N_i, c_i)/N_i$, which by Remark 4.1 equals $e(v_i/v_{i-1})$. Since subtracting integers preserves shortest 1-paths, we subtract $N_i\beta_i - 1$ from each entry, and we view our path as the shortest 1-path from 1 to $b'/e(v_i/v_{i-1})$, where $0 < b' \leq e(v_i/v_{i-1})$. By Corollary 5.11(i), the length of this path is at most $e(v_i/v_{i-1}) - b' + 1 \leq e(v_i/v_{i-1})$. This proves (i). Furthermore, if $i = 1$ and $\lambda_1 = 3/c$ with $c \geq 7$ and $c \equiv 1 \pmod 3$, then we can use Corollary 5.11(ii) instead to get a bound of $e(v_1/v_0) - 3 = e(v_1/v_0) - b'$.

To prove part (ii), we start by bounding the number of $w_{i-1,\lambda}$'s. By Proposition 5.12, these form the shortest N_i-path from λ_i to $e(v_{i-1}/v_{i-2})\lambda_{i-1}$ (excluding the endpoints). Since $N_i = e(v_{i-1}/v_0)$ by Corollary 4.4, $N_i\lambda_{i-1} \in \mathbb{Z}$ and $\lambda_i > e(v_{i-1}/v_{i-2})\lambda_{i-1}$ by Remark 5.14, it follows that $N_i\lambda_i > \lfloor N_i\lambda_i \rfloor \geq N_i e(v_{i-1}/v_{i-2})\lambda_{i-1}$. Again, using [OW18, Lemma A.7], this has the same length as the shortest 1-path from $N_i\lambda_i$ to $N_i e(v_{i-1}/v_{i-2})\lambda_{i-1}$ without the endpoints. By [OW18, Corollary A.12], this path consists of the integers from $\lfloor N_i\lambda_i \rfloor$ to $N_i e(v_{i-1}/v_{i-2})\lambda_{i-1}$, concatenated to the shortest 1-path P from $N_i\lambda_i$ to $\lfloor N_i\lambda_i \rfloor$. Since subtracting integers preserves shortest 1-paths, P has the same length as the shortest 1-path from $b'/e(v_i/v_{i-1})$ to 0, where b' is as in the previous paragraph. By Corollary 5.11(i), the length of P is at most $b' + 1$. We can improve this $b' + 1$ to b' if $i = 1$ and $\lambda_1 = 3/c$ (i.e., $b' = 3$) with $c \geq 5$ and $c \equiv 2 \pmod 3$ using Corollary 5.11(ii) instead. Putting everything together, the total number of valuations in part (ii) is

$$\underbrace{(e(v_i/v_{i-1}) - b' + 1)}_{\max \#\text{ of } v_{i,\lambda}} + \underbrace{(b'+1)}_{\max\ \text{size of } P} + \underbrace{(\lfloor N_i\lambda_i \rfloor - N_i(e(v_{i-1}/v_{i-2})\lambda_{i-1}))}_{\text{other } w_{i-1,\lambda}} - \underbrace{2}_{\text{endpoints}}. \tag{5.18}$$

Adding everything up gives the general bound in (ii). If $i = 1$ and $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, then either the first or the second term in (5.18) can be decreased by 1, depending on whether $c \equiv 1$ or $2 \pmod 3$.

If we are in case (iii), then there are no $v_{n,\lambda}$'s and P is the path consisting only of $N_n\lambda_n$ (which has only one endpoint). So the maximum number of components is $\lfloor N_n\lambda_n \rfloor - N_n e(v_{n-1}/v_{n-2})\lambda_{n-1} = N_n(\lambda_n - e(v_{n-1}/v_{n-2})\lambda_{n-1})$.

Part (iv) follows from Corollary 5.16(i). □

5.3. **Other regular models of $\mathbb{P}^1_K$.** In order to make the model of $\mathbb{P}^1_K$ that realizes the conductor-discriminant inequality, we have to work with certain contractions of the regular model associated to a Mac Lane valuation described in Proposition 5.12. In Proposition 5.25(i) and (iii), we describe this contraction and in §8, §9, and §10 we will use it to prove the conductor-discriminant inequality.

First, some setup. Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$. Let $N_n = \mathrm{lcm}(c_1, \ldots, c_{n-1}) = \deg(\varphi_n)$ (Corollary 4.4). We assume that $n \geq 1$ and $\lambda_n \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. Set $\lambda_0 = \lfloor \lambda_1 \rfloor$.

Let $\mathcal{Y}_v$ be the v-model of $\mathbb{P}^1_K$, and let $\mathcal{Y}_v^{\mathrm{reg}}$ be its minimal regular resolution. By Proposition 5.12, the following two Mac Lane valuations are included in $\mathcal{Y}_v^{\mathrm{reg}}$:

- $v' := w_{n-1,\lambda'} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v'_n(\varphi_n) = \lambda']$

- $v'' := v_{n,\lambda''} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v''_n(\varphi_n) = \lambda'']$,

where λ' is the entry directly following λ_n in the shortest N_n-path from λ_n to $e(v_{n-1}/v_{n-2})\lambda_{n-1}$ and λ'' is the entry directly preceeding λ_n in the shortest N_n-path from $\lceil N_n\lambda' \rceil / N_n$ to λ'. The valuations v' and v'' are respectively called the *successor* and *precursor* valuation to v. If $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$, we introduce the precursor valuation v^* of v', which may or may not be equal to v'' (but is never equal to v, see Remark 5.21):

- $v^* := v_{n,\lambda^*} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v^*_n(\varphi_n) = \lambda^*]$.

Lemma 5.19. *If $e(v_n/v_{n-1}) = 2$, then the valuations $v_{n,\lambda}$ from Proposition 5.12 are precisely v and v''.*

Proof. By Corollary 5.17(i), there are at most two valuations $v_{n,\lambda}$. By Proposition 5.12, the valuations v and v'' are included among the $v_{n,\lambda}$. So they are the only two. ☐

Since $\lambda_n \notin (1/N_n)\mathbb{Z}$, we have $\lfloor N_n\lambda_n \rfloor \leq N_n\lambda' < N_n\lambda_n < N_n\lambda'' \leq \lceil N_n\lambda_n \rceil$, the first inequality coming from [OW18, Corollary A.11]. Similarly, we have $N_n\lambda' < N_n\lambda^* \leq \lceil N_n\lambda_n \rceil$ when $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. Write $\tilde{\lambda}_n$ (resp. $\tilde{\lambda}'$, $\tilde{\lambda}''$, $\tilde{\lambda}^*$) for $N_n\lambda_n - \lfloor N_n\lambda_n \rfloor$ (resp. $N_n\lambda' - \lfloor N_n\lambda_n \rfloor$, $N_n\lambda'' - \lfloor N_n\lambda_n \rfloor$, $N_n\lambda^* - \lfloor N_n\lambda_n \rfloor$). Then we obtain

$$0 \leq \tilde{\lambda}' < \tilde{\lambda}_n < \tilde{\lambda}'' \leq 1.$$

Similarly, when $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$, we have

$$0 \leq \tilde{\lambda}' < \tilde{\lambda}^* \leq 1.$$

Proposition 5.20. *For any statement involving $\tilde{\lambda}^*$ below, assume $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$.*

(i) *The number $\tilde{\lambda}'$ immediately follows $\tilde{\lambda}_n$ in the shortest 1-path from $\tilde{\lambda}_n$ to 0.*

(ii) *The number $\tilde{\lambda}''$ immediately preceeds $\tilde{\lambda}_n$ in the shortest 1-path from 1 to $\tilde{\lambda}_n$.*

(iii) *The number $\tilde{\lambda}_n$ is the mediant of $\tilde{\lambda}'$ and $\tilde{\lambda}''$.*

(iv) *The number $\tilde{\lambda}^*$ immediately preceeds $\tilde{\lambda}'$ in the shortest 1-path from 1 to $\tilde{\lambda}'$.*

(v) *We have $\tilde{\lambda}^* \geq \tilde{\lambda}''$.*

(vi) *If e, e', e'', and e^* are the denominators of $\tilde{\lambda}_n$, $\tilde{\lambda}'$, $\tilde{\lambda}''$, and $\tilde{\lambda}^*$, respectively, then $e = e' + e''$ and $e' \mid (e - e^*)$.*

Proof. By [OW18, Lemma A.7], $N_n\lambda''$ immediately preceeds $N_n\lambda_n$ in the shortest 1-path from $\lceil N_n\lambda_n \rceil$ to λ_n. Also by [OW18, Lemma A.7], $N_n\lambda'$ immediately follows $N_n\lambda_n$ in the shortest 1-path from $N_n\lambda_n$ to $N_n e(v_i/v_{i-1})\lambda_{n-1}$, and thus in the shortest 1-path from $N_n\lambda_n$ to $\lfloor N_n\lambda_n \rfloor$ by [OW18, Lemma A.11]. Since translating by an integer preserves shortest 1-paths, subtracting $\lfloor N_n\lambda_n \rfloor$ from all entries of these paths yields parts (i) and (ii). Part (iii) follows from parts (i), (ii), and Proposition 5.9. The proof of part (iv) is exactly the same as that of part (ii), using λ^* and λ' instead of λ'' and λ_n.

Taking the shortest 1-path from 1 to $\tilde{\lambda}_n$ and then appending $\tilde{\lambda}'$ yields a 1-path P from 1 to $\tilde{\lambda}'$ by part (i). By part (iv), $\tilde{\lambda}^*$ lies somewhere in this path. In fact, by Proposition 5.9(i) applied to $\tilde{\lambda}'$ with $i = 0$, we see that $e^* < e'$, and that $\tilde{\lambda}^*$ is the entry in P closest to $\tilde{\lambda}'$ with denominator less than e'. Now, by part (i) and Proposition 5.9(ii) applied to e, we have $e' < e$. So $\tilde{\lambda}^* \neq \tilde{\lambda}_n$. Since $\tilde{\lambda}''$ immediately preceeds $\tilde{\lambda}_n$ in P, we conclude that $\tilde{\lambda}^* \geq \tilde{\lambda}''$, proving (v).

The first statement in part (vi) follows immediately from part (iii). The second follows from Proposition 5.9(iii) applied to the segment of P between $\tilde{\lambda}^*$ and $\tilde{\lambda}'$, where $c_1' = e'$, $c_0 = e$, and $c_i = e^*$ in that proposition. ☐

Remark 5.21. As a byproduct of the proof of Proposition 5.20, we see that $\lfloor N_n\lambda_n \rfloor \leq N_n\lambda'$. So

$$\lfloor N_n\lambda_n \rfloor \leq N_n\lambda' < N_n\lambda_n < N_n\lambda'' \leq N_n\lambda^* \leq \lceil N_n\lambda_n \rceil,$$

where the inequalities involving λ^* make sense only when $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. Furthermore, $N_n\lambda'$ is adjacent to $N_n\lambda_n$ in a 1-path, and the same is true for $N_n\lambda''$.

Example 5.22. In Example 5.10, if $\tilde{\lambda}_n = 3/8$, we would have $\tilde{\lambda}' = 1/3$, $\tilde{\lambda}'' = 2/5$, and $\tilde{\lambda}^* = 1/2$.

Corollary 5.23. *Let e, e', e'', and e^* be as in Proposition 5.20(vi). Then*

(i) $\lambda_n - \lambda' = 1/(N_n e e')$,
(ii) $\lambda'' - \lambda_n = 1/(N_n e e'')$,
(iii) $\lambda^* - \lambda' = 1/(N_n e^* e')$.
(iv) $\lambda'' - \lambda' = 1/(N_n e' e'')$.

Proof. By construction, the numbers e, e', e'', and e^* are the denominators of $N_n\lambda_n$, $N_n\lambda'$, $N_n\lambda''$, and $N_n\lambda^*$ respectively. By Remark 5.21 and the definition of 1-path, we have $N_n\lambda_n - N_n\lambda' = 1/(ee')$, $N_n\lambda'' - N_n\lambda_n = 1/(ee'')$, and $N_n\lambda^* - N_n\lambda' = 1/(e^*e')$. This proves (i), (ii), and (iii). Equalities (i) and (ii), along with Proposition 5.20(vi), imply (iv). ☐

Lemma 5.24. *Let v, v' and v'' be as above, and let v^* be as above if $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. If e, e', e'', and e^* are defined as in Proposition 5.20(vi), then $e = e(v/v_{n-1})$, $e' = e(v'/v_{n-1})$, $e'' = e(v''/v_{n-1})$ and $e^* = e(v^*/v_{n-1})$.*

Proof. By construction, e is the denominator of $N_n\lambda_n$ (and similarly for e', e'', and e^*). By [OW18, Lemma 5.3(ii)], $e(v/v_0) = \mathrm{lcm}(N_n, c_n)$, where c_n is the denominator of λ_n. By [OW18, Lemma A.6], this is equal to $N_n e$. Since $N_n = e(v_{n-1}/v_0)$, we have $e = e(v/v_{n-1})$. This proves the lemma for e, and the proofs for e', e'', and e^* are identical. ☐

Proposition 5.25. *Assume the setup so far in §5.3. Recall that $\mathcal{Y}_v^{\mathrm{reg}}$ is the minimal regular resolution of the v-model of $\mathbb{P}^1_K$.*

(i) *If $\mathcal{Y}_v^{\mathrm{reg}} \to \mathcal{Z}$ is the morphism contracting the v-component of $\mathcal{Y}_v^{\mathrm{reg}}$, then $\mathcal{Z}$ is also a regular model of $\mathbb{P}^1_K$.*
(ii) *The minimal regular resolution $\mathcal{Y}_{v'}^{\mathrm{reg}}$ of the v'-model of $\mathbb{P}^1_K$ is a (possibly trivial) contraction of $\mathcal{Z}$.*
(iii) *Suppose $\lambda' \in \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. Then the canonical contraction $\mathcal{Y}_v^{\mathrm{reg}} \to \mathcal{Y}_{v'}^{\mathrm{reg}}$ is the morphism contracting all $v_{n,\lambda}$ as in Proposition 5.12.*

Proof. We prove (i) first. It suffices to show that the v-component $\overline{Y}_v$ of $\mathcal{Y}_v^{\mathrm{reg}}$ is a -1-component. The two components that $\overline{Y}_v$ intersects are the v' and v''-components $\overline{Y}_{v'}$ and $\overline{Y}_{v''}$, respectively, and the intersection is transverse. By [Liu02, Proposition 9.1.21], it suffices to show that the multiplicity of $\overline{Y}_v$ is the sum of the multiplicities of $\overline{Y}_{v'}$ and $\overline{Y}_{v''}$, i.e., that $e(v/v_0) = e(v'/v_0) + e(v''/v_0)$. To prove this, we can replace v_0 with v_{n-1}, and now the equality follows from Proposition 5.20(vi) and Lemma 5.24.

Part (ii) follows immediately from part (i), since $\mathcal{Z}$ includes v'. Part (iii) follows easily by applying Proposition 5.12 to v', since there will be no $v_{n,\lambda}$'s in this resolution by our assumption that $\lambda' \in \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. ☐

5.4. **The Mac Lane valuation associated to a polynomial.** Let $\alpha \in \mathcal{O}_{\overline{K}}$ such that $\nu_K(\alpha) > 0$ and the minimal polynomial of α has degree at least 2. In §5.4, we construct Mac Lane valuations associated to f that will be used in §6 and throughout the paper.

Write

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$$

for the unique Mac Lane valuation on $K(x)$ over which f is a proper key polynomial (Proposition 4.9(iv)). As usual, write $v_0, v_1, \ldots, v_n = v_f$ for the intermediate valuations. For $1 \leq i \leq n$, write $\lambda_i = b_i/c_i$ in lowest terms. Let $N_i = \operatorname{lcm}(c_1, \ldots, c_{i-1}) = \deg(\varphi_i)$ (Corollary 4.4).

Remark 5.26. If the roots of f generate a tame extension, it is easy to read off the polynomials φ_i and integers λ_i from the truncations of Newton-Puiseux expansions of the roots of f with respect to some choice of uniformizer t, as we now explain. Using Proposition 4.9(iii), we see that we can take φ_i to be the minimal polynomials of the truncations of the Newton-Puiseux expansions just before there is a jump in the lcm of the denominators of the exponents in the expansion. If α is a root of f, then Corollary 4.12 shows that $\lambda_i = \nu_K(\varphi_i(\alpha)) = \sum_{\varphi_i(\beta)=0} \nu_K(\alpha - \beta)$. If $\deg(\varphi_i) = m$, then the Galois group of the splitting field of the tame extension generated by the roots of φ_i is generated by the automorphism $t^{1/m} \mapsto \zeta_m t^{1/m}$ for a primitive m^{th} root of unity ζ_m. Since the induced $\mathbb{Z}/m\mathbb{Z}$-action on the roots of φ_i is transitive, a direct computation then shows that for each root β of φ_i, the quantity $\nu_K(\alpha - \beta)$ is equal to one of the the exponents in α where the lcm of the denominators of the exponents jumps. (This is the content of [Sri19, Lemma 8.13] using the language of characteristic/jump exponents.)

For example, let $K = \mathbb{C}((t))$ and let f is the minimal polynomial of $2t - t^{5/2} + t^{8/3} - 3t^{7/2} + t^{23/6}$. Then v_f has the form

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1, v_1(\varphi_2) = \lambda_2, v_1(\varphi_3) = \lambda_3],$$

and we can take $\varphi_1 = x - 2t$, φ_2 to be the minimal polynomial of $2t - t^{5/2}$ and φ_3 to be the minimal polynomial of $2t - t^{5/2} + t^{8/3}$, and $\lambda_1 = 5/2, \lambda_2 = 5/2 + 8/3, \lambda_3 = 7/2 + 2(8/3) + 3(5/2)$. This example also shows that $\deg(\varphi_i)$ and the invariants λ_i contain the same information as the characteristic exponents of the Newton-Puiseux expansion of a root of f as in [Sri19, Example 8.22] in the tame case.

Lemma 5.27.

(i) *If a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ includes v_f, then D_α intersects only one irreducible component of the special fiber of $\mathcal{Y}$.*

(ii) *If the model $\mathcal{Y}$ is the minimal regular resolution of the v_f-model of $\mathbb{P}^1_K$, then the irreducible component from part (i) is the one corresponding to v_f.*

Proof. The multiplicity of the v_f-component of $\mathcal{Y}$ in the special fiber is $e(v_n/v_{n-1})e(v_{n-1}/v_0)$, which is $e(v_n/v_{n-1})N_n$ by Remark 4.1. But $e(v_n/v_{n-1}) = \deg(f)/\deg(\varphi_n)$ by Lemma 4.3(iii) and $N_n = \deg(\varphi_n)$. So the multiplicity is equal to $\deg(f)$.

By Corollary 4.12, $v_f(\varphi_n(\alpha)) = \lambda_n$. So by [OW18, Lemma 7.3(iii)] and Proposition 5.2, D_α intersects a regular point z on the v_f-model of $\mathbb{P}^1$, which is also a smooth point of the reduced special fiber by [OW18, Lemma 7.1]. By the previous line, we conclude that the point z is not part of the base locus of the natural map from the *minimal* regular resolution of the v_f-model to the v_f-model, and this proves part (ii). Now, suppose we blow up a closed point on the special fiber of this model. If the point is z, then [LL99, Lemma 1.4(a)] shows that the exceptional divisor E has multiplicity $\deg(f)$ as well, and [LL99, Lemma 5.1(a)] shows that D_α intersects a single component of this new model (which is necessarily E), at a regular point that is also a smooth point of the reduced special fiber.

On the other hand, if the point is not z, then D_α intersects the inverse image of the point z, which is a regular point of the new model that is a smooth point of the reduced special fiber (and that lies on a component of multiplicity $\deg(f)$). By induction, after making any number of closed point blowups, D_α intersects a unique component of multiplicity $\deg(f)$. Since $\mathcal{Y}$ is attained from the v_f-model by successive closed point blowups, we are done with part (i). ☐

Lemma 5.28. *If* $[v_0, v_1(\varphi_1) = \lambda_1, \dots, v_n(\varphi_n) = \lambda_n] = v_f$ *for some* f*, then for* $1 \leq i \leq n$*, we have* $\lambda_i \notin \Gamma_{v_{i-1}} = (1/N_i)\mathbb{Z}$*.*

Proof. If $\lambda_i \in \Gamma_{v_{i-1}}$, then $e(v_i/v_{i-1}) = 1$. If $i = n$, applying Lemma 4.3(iii) to v_n, contradicts the fact that $\deg(f) > \deg(\varphi_n)$. For $i < n$, applying Lemma 4.3(iii) to v_i contradicts the fact that $\deg(\varphi_{i+1}) > \deg(\varphi_i)$. ☐

Lemma 5.29. *In the situation of Lemma 5.28, if* $\deg(f)/\deg(\varphi_n) = 2$*, then* $\deg(\varphi_n)\lambda_n$ *is a half-integer.*

Proof. By Lemma 4.3(iv), $\deg(f)\lambda_n \in \mathbb{Z}$. Since $N_n = \varphi_n$ by Corollary 4.4, Lemma 5.28 shows that $\deg(\varphi_n)\lambda_n \notin \mathbb{Z}$. So $\deg(\varphi_n)\lambda_n$ is a half-integer. ☐

For the rest of this section, as well as §6, we will use the following notation.

Notation 5.30. Since Lemma 5.28 is true, we are in the situation of §5.3. So define

- $v'_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \dots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v'_n(\varphi_n) = \lambda']$
- $v''_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \dots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v''_n(\varphi_n) = \lambda'']$,

to be the successor and precursor valuations to v_f. If $\lambda' \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$, then we also define

- $v^*_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \dots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v^*_n(\varphi_n) = \lambda^*]$

to be the precursor valuation to v'_f.

Remark 5.31. If f is Eisenstein, then f is a proper key polynomial over $[v_0, v_1(x) = 1/\deg(f)]$. Since $1/\deg(f) > 0$ is a shortest 1-path and $[v_0, v_1(x) = 0] = v_0$, we have $v'_f = v_0$. Furthermore, any key polynomial over a valuation of the form $[v_0, v_1(x) = 1/d]$ is Eisenstein of degree d by Lemma 4.3(ii) and (iii).

The following lemmas are used only in §10, but they are convenient to state here.

Lemma 5.32. *Let* λ_n*,* λ'*,* λ''*, and* v_f *be as above. Suppose* $g \in K[x]$ *such that* $v_g = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \mu_n]$ *is the unique Mac Lane valuation over which* g *is a proper key polynomial. For any* μ*, write* $v_\mu = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_{n-1}(\varphi_{n-1}) =$

λ_{n-1}, $v_n(\varphi_n) = \mu]$, *and define* μ' *and* μ'' *so that* $v_{\mu'}$ *and* $v_{\mu''}$ *are the successor and precursor valuation to* v_g, *respectively. If* $\lambda_n < \mu_n$, *then either* $\lambda'' \leq \mu_n$ *or* $\lambda_n \leq \mu'$.

Proof. Let $M = \lfloor N_n\lambda_n \rfloor$. Let $\tilde{\lambda}_n = N_n\lambda_n - M$, let $\tilde{\mu}_n = N_n\mu_n - M$, and similarly for λ', λ', μ', and μ''. This is consistent with the notation leading up to Proposition 5.20.

Suppose the denominator of $\tilde{\lambda}_n$ is less than or equal to the denominator of $\tilde{\mu}_n$. By Propositions 5.20(i) and 5.9(ii), $\tilde{\mu}'$ is the entry immediately preceeding $\tilde{\mu}_n$ in the Farey sequence with denominator bounded by that of $\tilde{\mu}_n$. Since $\tilde{\lambda}_n$ preceeds $\tilde{\mu}_n$ in the same Farey sequence, we have $\tilde{\lambda}_n \leq \tilde{\mu}'$, which implies $\lambda_n \leq \mu'$.

On the other hand, suppose the denominator of $\tilde{\mu}_n$ is less than or equal to the denominator of $\tilde{\lambda}_n$. By Propositions 5.20(ii) and 5.9(i), $\tilde{\lambda}''$ is the entry immediately following $\tilde{\lambda}_n$ in the Farey sequence with denominator bounded by that of $\tilde{\lambda}_n$. Since $\tilde{\mu}_n$ follows $\tilde{\lambda}_n$ in the same Farey sequence, we have $\tilde{\lambda}'' \leq \tilde{\mu}_n$, which implies $\lambda'' \leq \mu_n$. ☐

Lemma 5.33. *Let* $f = f_1f_2$ *with* f_1 *and* f_2 *monic irreducible in* $\mathcal{O}_K[x]$ *with roots* α_1 *and* α_2, *respectively, of positive valuation. Suppose we are in one of the following three cases:*

(i) $\nu_K(\alpha_1) = 2/c$ *with* $c \geq 3$ *odd, and* f_2 *is linear.*

(ii) $v_{f_1} = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ *with* $\deg(f_1)/\deg(\varphi_n) = 2$, *the polynomial* f_1 *is not Eisenstein, and* f_2 *is linear. Furthermore, there does not exist* $\gamma \in K$ *such that* $\nu_K(\alpha_1 - \gamma) = a/2$ *with* $a \geq 3$ *an odd integer, and* $\nu_K(\alpha_2 - \gamma) > a/2$.

(iii) *The polynomial* f_1 *is as in part (ii) and also has degree* 2, *and* f_2 *is Eisenstein.*

Then $v_{f_1}(f) \in 2\Gamma_{v_{f_1}}$.

Proof. In case (i), we have $v_{f_1} = [v_0, v_1(x) = 2/c]$ and thus $\Gamma_{v_{f_1}} = (1/c)\mathbb{Z}$. By Lemma 4.3(ii), $v_{f_1}(f_1) = c(2/c) = 2$. Since $f_2 = x - \alpha_2$ for some $\alpha_2 \in \mathcal{O}_K$ with $\nu_K(\alpha_2) \geq 1$, we have $v_{f_1}(f_2) = 2/c$. So $v_{f_1}(f) = 2 + 2/c \in (2/c)\mathbb{Z}$. This finishes case (i).

In cases (ii) and (iii), we write $v_n = v_{f_1}$. Observe that, by Lemma 4.3(iii), $e(v_n/v_{n-1}) = 2$. So $\Gamma_{v_{n-1}} = 2\Gamma_{v_n}$. By Lemma 4.3(ii) and (iii), $v_n(f_1) = 2\lambda_n \in 2\Gamma_{v_n}$.

If f_2 is linear, write $f_2 = \varphi_1 - \beta$, where $\beta \in \mathcal{O}_K$ has positive valuation. Then $v_{f_1}(f_2) = \min(\lambda_1, \nu_K(\beta))$. If this minimum equals $\nu_K(\beta)$ (which is an integer), then $v_{f_1}(f_2) \in \mathbb{Z} \subseteq \Gamma_{v_{n-1}} = 2\Gamma_{v_n}$. If this minimum equals λ_1, then since $\lambda_1 \in \Gamma_{v_{n-1}}$ as long as $n \geq 2$, we have $v_{f_1}(f_2) \in \Gamma_{v_{n-1}} = 2\Gamma_{v_n}$ as long as $n \geq 2$. If this minimum equals λ_1 and $n = 1$, then $\deg(f_1) = 2$ and therefore $\nu_K(\alpha_1) = a/2$ for some odd integer a, which in turn implies that $\lambda_1 = \nu_K(\varphi_1(\alpha_1)) = a/2$ by Corollary 4.12 and the fact that φ_1 is linear. Since f_1 is not Eisenstein, $a \geq 3$. Also, $\varphi_1(\alpha_2) = \beta$ and $\nu_K(\beta) > \lambda_1 = a/2$ by assumption. Writing $\varphi_1 = x - \gamma$, we find ourselves in the case that is ruled out at the end of case (ii). This finishes case (ii).

In case (iii), then $v_{f_1}(f_1) \in 2\Gamma_{v_{f_1}}$ as in part (ii). Now, by Lemma 4.3(iii), $v_{f_1} = [v_0, v_1(\varphi_1) = \lambda_1]$, with $\lambda_1 = a/2$ for a odd. Again, f_1 is not Eisenstein, so $a \geq 3$. If f_2 is Eisenstein, then when f_2 is written as a polynomial in φ_1, Lemma 4.2 shows that the constant term a_0 has valuation 1. Since $v_{f_1}(\varphi_1) = \lambda_1 > 1$, we have $v_{f_1}(f_2) = 1$. Since $1 \in \Gamma_{v_{n-1}} = 2\Gamma_{v_n}$, we are done. ☐

6. Regular horizontal divisors

Let $\alpha \in \mathcal{O}_{\overline{K}}$ such that $\nu_K(\alpha) > 0$ and the minimal polynomial $f(x) \in K[x]$ of α has degree at least 2. The main goal of this section is to construct a regular model of $\mathbb{P}^1_K$ on which the horizontal divisor D_α (§5.1) is regular. This model is called $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$, and is defined in Definition 6.1. The regularity statement about the horizontal divisor is Theorem 6.9. For later purposes, it will be more useful to consider the minimal regular modification $\mathcal{Y}'_{f,0}$ of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ that includes v_0. Proposition 6.20 gives an upper bound on the number of irreducible components of the special fiber of $\mathcal{Y}'_{f,0}$.

Throughout §6, we use Notation 5.30. So v_f is the unique Mac Lane valuation over which f is a key polynomial, and v'_f, v''_f, and v^*_f are the related Mac Lane valuations from Notation 5.30. For simplicity, we write $e = e(v_f/v_{n-1})$, $e' = e(v'_f/v_{n-1})$, $e'' = e(v''_f/v_{n-1})$, and $e^* = e(v^*_f/v_{n-1})$. This is consistent with the notation in Lemma 5.24 and Proposition 5.20(vi). We record for later usage that $e = \deg(f)/\deg(\varphi_n)$ by Lemma 4.3(iii).

6.1. **The model $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$.** We now define the regular models of $\mathbb{P}^1_K$ that will be the focus of the rest of §6, and we state some specialization properties that will be useful for constructing our desired regular model in which D_α is regular.

Definition 6.1. If v is a Mac Lane valuation, then $\mathcal{Y}^{\mathrm{reg}}_v$ is the minimal regular resolution of the v-model of $\mathbb{P}^1_K$. The model $\mathcal{Y}'_{f,0}$ (resp. $\mathcal{Y}_{f,0}$) is the minimal regular resolution of the $\{v'_f, v_0\}$-model of $\mathbb{P}^1_K$ (resp. the $\{v_f, v_0\}$-model), see Corollary 5.16. The map $\mathcal{Y}^{\mathrm{reg}}_{v_f} \to \mathcal{Z}_f$ (resp., $\mathcal{Y}_{f,0} \to \mathcal{Z}_{f,0}$) is the contraction of the v_f-component of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ (resp. $\mathcal{Y}_{f,0}$).

Remark 6.2. Proposition 5.25(i) shows that $\mathcal{Z}_f$ and $\mathcal{Z}_{f,0}$ are regular, and Proposition 5.25(ii) shows that $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ is a contraction of $\mathcal{Z}_f$ (resp. $\mathcal{Y}'_{f,0}$ is a contraction of $\mathcal{Z}_{f,0}$). By Proposition 5.12, the valuations v'_f and v''_f are included in $\mathcal{Y}_f$ and $\mathcal{Y}_{f,0}$, and thus in $\mathcal{Z}_f$ and $\mathcal{Z}_{f,0}$. Similarly, observe that by Proposition 5.12 and the construction of v^*, the valuation v^* is included in $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ when $\lambda' \notin \Gamma_{v_{n-1}}$. Thus we have the following commutative diagram of dominant maps between models:

$$\begin{array}{ccccc} \mathcal{Y}_{f,0} & \longrightarrow & \mathcal{Z}_{f,0} & \longrightarrow & \mathcal{Y}'_{f,0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{Y}^{\mathrm{reg}}_{v_f} & \longrightarrow & \mathcal{Z}_f & \longrightarrow & \mathcal{Y}^{\mathrm{reg}}_{v'_f} \end{array}$$

Proposition 6.3. *Let α and f be as in this section, and let $\mathcal{Y}^{\mathrm{reg}}_{v_f}$, $\mathcal{Y}_{f,0}$, $\mathcal{Z}_f$, $\mathcal{Z}_{f,0}$, $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$, and $\mathcal{Y}'_{f,0}$, be as in Definition 6.1.*

(i) *On $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ and $\mathcal{Y}_{f,0}$, the divisor D_α intersects only the v_f-component of the special fiber.*
(ii) *On $\mathcal{Z}_f$ and $\mathcal{Z}_{f,0}$, the divisor D_α meets the intersection of the two components of the special fiber corresponding to v'_f and v''_f.*
(iii) *If $\lambda' \notin \Gamma_{v_{n-1}}$, then on $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ and $\mathcal{Y}'_{f,0}$, the divisor D_α meets the intersection of the two components of the special fiber corresponding to v'_f and v^*_f.*
(iv) *If $\lambda' \in \Gamma_{v_{n-1}}$, then on $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ and $\mathcal{Y}'_{f,0}$, the divisor D_α intersects only the v'_f-component of the special fiber.*

Proof. Part (i) for $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ is just Lemma 5.27, and the proof for $\mathcal{Y}_{f,0}$ follows because none of the extra components on the special fiber of $\mathcal{Y}_{f,0}$ intersect the v_f-component. Since $\mathcal{Z}_f$ is the contraction of the v_f-component of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$, and by Remark 5.15 we have that this component intersects the v'_f and v''_f-components of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$, it follows that D_α meets the intersection of these two components. The same proof works for $\mathcal{Z}_{f,0}$, proving part (ii).

In part (iii), we have $\lambda^* \geq \lambda''$ by Proposition 5.20(v). Furthermore, by Proposition 5.25(ii), $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ is a contraction of $\mathcal{Z}_f$. By Proposition 5.12 and Remark 5.15 and the definition of v^*_f, the v'_f and v^*_f-components of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ intersect. Looking at Figure 3 from Proposition 5.12 and using part (ii), one sees that D_α meets this intersection point. The proof for $\mathcal{Z}_{f,0}$ is the same.

Lastly, in part (iv), Proposition 5.25(iii) shows that, in the language of Proposition 5.12 all components corresponding to the $v_{n,\lambda}$ are contracted in the morphism $\mathcal{Y}^{\mathrm{reg}}_f \to \mathcal{Y}^{\mathrm{reg}}_{v'_f}$. From the dual graph diagram in Proposition 5.12 and part (ii), D_α meets only the v'_f-component. The proof for $\mathcal{Y}'_{f,0}$ is the same. □

Corollary 6.4. *For $1 \leq i \leq n$, let α_i be a root of φ_i. If $\lambda' \notin \Gamma_{v_{n-1}}$, then D_α and D_{α_i} do not meet on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ for any i. If $\lambda' \in \Gamma_{v_{n-1}}$, then D_α and D_{α_n} meet on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$, but D_α does not meet D_{α_i} for any i between 1 and $n-1$.*

Proof. For $1 \leq i \leq n-1$, the φ_n-adic expansion of φ_i is just φ_i itself, so applying Corollary 4.13 to v'_f and φ_i shows that $\nu_K(\varphi_n(\alpha_i)) \leq \lambda'$. Since α_n is a root of φ_n, we have $\nu_K(\varphi_n(\alpha_n)) = \infty$.

Assume $\lambda' \notin \Gamma_{v_{n-1}}$. From the previous paragraph, $\nu_K(\varphi_n(\alpha_n))$ does not lie between λ' and λ^* for any i from 1 to n. As a consequence, Corollary 5.4 and Proposition 6.3(iii) show that D_α does not meet D_{α_i} on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$.

Now assume $\lambda' \in \Gamma_{v_{n-1}}$. If $1 \leq i \leq n-1$, then since $\nu_K(\varphi_n(\alpha)) = \lambda_n > \lambda'$ (Corollary 4.12) and $\nu_K(\varphi_n(\alpha_i)) < \lambda'$, Corollary 5.3 shows that D_α does not meet D_{α_i} on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$. Since both $\nu_K(\varphi_n(\alpha)) = \lambda_n$ and $\nu_K(\varphi_n(\alpha_n)) = \infty$ are both greater than λ', Proposition 5.2 shows that they meet on the special fiber of the v'_f-model of $\mathbb{P}^1_K$. By [OW18, Lemma 7.3(i)], the point z where they meet is a regular point of the v'_f-model, and thus the natural map from $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ to the v'_f-model is an isomorphism in a neighbourhood of z, which in turn implies that these two components still meet on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$. □

Lemma 6.5. *Let $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ be the φ_n-adic expansion of f.*

(i) *We have $v'_f(f - \varphi_n^e) = e\lambda_n = v_f(f) = v_f(a_0) = v'_f(a_0)$.*
(ii) *We have $v'_f(f) = ev'_f(\varphi_n) = e\lambda'$.*
(iii) *If $\lambda_n \notin \Gamma_{v_{n-1}}$, then $v^*_f(f - \varphi_n^e) = v^*_f(f) = e\lambda_n$.*
(iv) *If β is a root of $f - \varphi_n^e$, then D_β does not meet D_α on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$.*

Proof. By Lemma 4.3(ii), we have $v_f(f) = v_f(\varphi_n^e) = v_f(a_0) = e\lambda_n$, and $v_f(a_i\varphi_n^i) \geq e\lambda_n$ for $1 \leq i \leq e-1$. It remains to prove the first equality in part (i). Now, $v'_f(a_i\varphi_n^i) = v_f(a_i\varphi_n^i) - i(\lambda_n - \lambda') \geq e\lambda_n - i(\lambda_n - \lambda')$ for $1 \leq i \leq e-1$. By Corollary 5.23(i), this equals $e\lambda_n - i/(N_n e' e)$, which is strictly greater than $e\lambda_n - 1/(N_n e')$. Since $1/(N_n e')$ generates $\Gamma_{v'_f}$, and $e\lambda_n = v_f(a_0) = v'_f(a_0) \in \Gamma_{v'_f}$, we in fact have that $v'_f(a_i\varphi_n^i) \geq e\lambda_n$ for $1 \leq i \leq e-1$. This proves part (i).

Also, note that φ_n^e is a term in the φ_n-adic expansion of f with minimal v_f-valuation, and it is also the term whose valuation is decreased the most when v_f is replaced with v'_f. So $v'_f(f) = v'_f(\varphi_n^e) = e\lambda'$. This proves part (ii).

For part (iii), one has that $v_f^*(a_i\varphi_n^i) > v_f(a_i\varphi_n^i) \geq v_f(a_0) = v_f^*(a_0)$ for $1 \leq i \leq e$, where we take $a_n = 1$. So $v_f^*(f) = v_f^*(a_0)$, and $v_f^*(f - \varphi_n^e) = v_f^*(a_0)$ as well. Since $v_f^*(a_0) = v_f(a_0) = v_f(f) = e\lambda_n$, by Lemma 4.3(ii), this proves (iii).

By part (i), Corollary 4.13 applies to to $f - \varphi_n^e$ and v'_f. Thus β, being a root of $f - \varphi_n^e$, has $\nu_K(\varphi_n(\beta)) \leq \lambda'$. If $\lambda_n \notin \Gamma_{v_{n-1}}$, then Proposition 6.3(iii) shows that D_α meets the special fiber of $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ at the intersection of the v'_f and v_f^*-components. By Corollary 5.4, the same is not true of D_β, proving part (iv) in this case. If, $\lambda_n \in \Gamma_{v_{n-1}}$, then Proposition 6.3(iv) shows that D_α meets the special fiber of $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ only on the v'_f-component. Since $\nu_K(\varphi_n(\alpha)) = \lambda_n > \lambda'$, (Corollary 4.12) while $\nu_K(\varphi_n(\beta)) \leq \lambda'$ as we have seen, Corollary 5.3 shows that D_α and D_β do not meet on the v'_f-model of $\mathbb{P}_K^1$. Thus they do not meet on $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$, finishing part (iv). ☐

Lemma 6.6. *Let $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ be as in Definition 6.1. Let $b = 0$ if $\lambda' \in \Gamma_{v_{n-1}}$ and $b = e^*$ if $\lambda \notin \Gamma_{v_{n-1}}$. Let $r = (e - b)/e'$. Recall that α is a root of f, and can be viewed as a point on the generic fiber of $\mathbb{P}_K^1$ using our chosen coordinate.*

(i) *We have $r \in \mathbb{N}$, and there is a monomial t in $\varphi_1, \varphi_2, \ldots, \varphi_{n-1}$ over K such that the horizontal divisor D_α obtained by taking the closure of the point α in $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ is locally cut out by the divisor of $t^r f/\varphi_n^b$.*

(ii) *Let $s = t^r$. Then $v'_f(sf/\varphi_n^b) = 0$. Furthermore, if $\lambda' \notin \Gamma_{v_{n-1}}$, then $v_f^*(sf/\varphi_n^b) = 0$ as well.*

(iii) *We have $v'_f(t\varphi_n^{e'}) = 0$.*

Proof. Let z be the point where D_α meets the special fiber of $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$. i.e., the specialization of $f(x) = 0$. The function f in general does not locally cut out D_α at z, because $\mathrm{div}(f)$ might also include vertical components passing through z. We construct t below so that $\mathrm{div}(t^r f/\varphi_n^b)$ no longer has these vertical components, and furthermore has the same horizontal component passing through z.

First assume $\lambda' \in \Gamma_{v_{n-1}}$. Then $e' = 1$ and $b = 0$, so $r = e$. By Proposition 6.3(iv), z lies on a unique component of the special fiber, namely the v'_f-component. Furthermore, since the value group of v'_f is $\Gamma_{v_{n-1}}$, and $\Gamma_{v_{n-1}}$ is generated by $v'_f(\varphi_1), v'_f(\varphi_2), \ldots, v'_f(\varphi_{n-1})$, one can find a monomial t in $\varphi_1, \ldots, \varphi_{n-1}$ such that $v'_f(t) = -v'_f(\varphi_n) = -\lambda'$. This proves (iii) in this case. Since $v'_f(f) = ev'_f(\varphi_n)$ (Lemma 6.5(ii)) and $s = t^r$, we have $v'_f(sf) = e\lambda' - r\lambda' = 0$. Note that for $1 \leq i \leq n-1$, Corollary 6.4 shows that the specialization of $\varphi_i(x) = 0$ on the special fiber of $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ is *not* the point z. Since $v'_f(sf) = 0$ and the divisor of s has no horizontal components intersecting z, the divisor of sf is locally what we seek. This proves (i) and (ii) in this case.

Now assume that $\lambda' \notin \Gamma_{v_{n-1}}$. Then $b = e^*$, and $r = (e - e^*)/e'$ is an integer by Proposition 5.20(vi). By Proposition 6.3(iii), z is the intersection of the v'_f and v_f^*-components of the special fiber. By Corollary 6.4, the specialization of $\varphi_i = 0$ is not z for any $1 \leq i \leq n$. So to prove parts (i) and (ii), it suffices to construct a monomial t in $\varphi_1, \ldots, \varphi_{n-1}$ such that $s = t^r$ and $v'_f(sf/\varphi_n^b) = v_f^*(sf/\varphi_n^b) = 0$. Furthermore, since $v'_f(f) = ev'_f(\varphi_n) = e\lambda'$ (Lemma 6.5(ii)), we have $v'_f(t\varphi_n^{e'}) = (1/r)v'_f(s\varphi_n^{re'}) = (1/r)v'_f(\varphi_n^{b+re'}/f) = (1/r)(b + re' - e)\lambda' = 0$, proving (iii).

Let us construct t. By Lemma 6.5(ii) and (iii), $v_f^*(f) - v_f'(f) = e(\lambda_n - \lambda')$. Using Corollary 5.23(iii), we obtain that $v_f^*(f) - v_f'(f) = e^*(\lambda^* - \lambda')$. So

$$v_f^*\left(\frac{f}{\varphi_n^{e^*}}\right) = v_f'\left(\frac{f}{\varphi_n^{e^*}}\right) = \lambda'(e - e^*) = r\lambda' e'. \tag{6.7}$$

Arguing as before, since $\lambda' e'$ lies in $\Gamma_{v_{n-1}}$, we can find a monomial t in $\varphi_1, \ldots, \varphi_{n-1}$ such that $v_f'(t) = -\lambda' e'$. Setting $s = t^r$, we have $v_f'(s) = -r\lambda' e' = -v_f'(f/\varphi_n^{e^*}) = -v_f^*(f/\varphi_n^{e^*})$. Since s is a monomial in $\varphi_1, \ldots, \varphi_{n-1}$, we have $v_f'(s) = v_f^*(s)$. So $v'(sf/\varphi_n^{e^*}) = v^*(sf/\varphi_n^{e^*}) = 0$, and $sf/\varphi_n^{e^*}$ locally gives D_α. □

The next theorem, proving that the divisor D_α is regular on $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$, is one of the most important in the paper, and it also has a rather technical proof. To aid the reader's understanding, we first sketch an example before proving the theorem.

Example 6.8. Let α be a root of $f(x) = x^8 + \pi_K x^6 + \pi_K^3$. Let z be the intersection of D_α with the special fiber of $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ as in Lemma 6.6. We will use the function sf/φ_n^b from Lemma 6.6 and another function g (which will come from a coordinate function along one of the components where z specializes) to cut out the maximal ideal in the local ring at z, which in turn shows that z is cut out by the principal ideal (g) in $\mathcal{O}_{D_\alpha, z}$, implying that D_α is regular.

One checks that f is a proper key polynomial over $v_f = [v_0,\, v_1(x) = 3/8]$. By Examples 5.10 and 5.22, $\lambda' = 1/3$ and $\lambda^* = 1/2$. So $e = 8$, $e^* = 2$, and $e' = 3$. The model $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ includes exactly the valuations $[v_0,\, v_1(x) = \lambda]$ for $\lambda \in \{0, 1/3, 1/2, 1\}$. In the notation of Lemma 6.6, $r = (e - e^*)/e' = 2$. Following the proof of Lemma 6.6 in this example, we can choose any $t \in K$ such that $v_f'(t) = -\lambda' e' = -1$. Specifically, we take $t = \pi_K^{-1}$ and thus $s = t^r = \pi_K^{-2}$. The horizontal divisor D_α is locally cut out by sf/x^2, or $f/(\pi_K^2 x^2)$. We claim that the maximal ideal of z in $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ is generated by $f/(\pi_K^2 x^2)$ and x^3/π_K. As remarked above, this shows that D_α is regular on $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ in this example.

To prove the claim, first note that, by Proposition 6.3(iii), the point z lies at the intersection of the v_f' and v_f^*-components. Let $\eta = f/(\pi_K^2 x^2) - x^8/(\pi_K^2 x^2)$ (we subtract the leading term from f). Then

$$\eta = \frac{x^4}{\pi_K} + \frac{\pi_K}{x^2}.$$

It is easy to check using the definition of a Mac Lane valuation that $v_f'(\eta) = 1/3$ and $v_f^*(\eta) = 0$ (for both valuations, the dominant term is π_K/x^2). One can also check that the horizontal part of the divisor of η does not intersect z. Since $v_f'(\eta)$ generates $\Gamma_{v_f'} = (1/3)\mathbb{Z}$, the divisor of η cuts out the v_f'-component. This component is locally isomorphic to $\mathbb{A}_k^1$ and by [Rüt14, Lemma 4.29], a local coordinate on this component is given by the residual image of x^3/π_K. Since z corresponds to $x^3/\pi_K = 0$, the maximal ideal of the local ring at z is generated by η and x^3/π_K. Since $x^8/(\pi^2 x^2) = (x^3/\pi_K)^2$, the ideal $(\eta, x^3/\pi_K)$ equals $(f/(\pi^2 x^2), x^3/\pi_K)$.

The proof of Theorem 6.9 below generalizes the strategy from Example 6.8 to all α.

Theorem 6.9. *Let $\alpha \notin K$ be algebraic over K such that $\nu_K(\alpha) > 0$, and let f be its minimal monic polynomial. Let v_f' be the successor valuation to v_f as above. Suppose $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ is as in Definition 6.1. Then D_α is regular on $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$.*

Proof. Let $z \in \mathcal{Y}_{v_f'}^{\mathrm{reg}}$ be the point where D_α intersects the special fiber. In light of Lemma 6.6, it suffices to show that z is a regular point of $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ with system of parameters containing $t^r f/\varphi_n^b$, where t, r, and b are defined as in Lemma 6.6. Specifically, $b = e^*$ if $\lambda' \notin \Gamma_{v_{n-1}}$, and $b = 0$ if $\lambda' \in \Gamma_{v_{n-1}}$. In both cases, $r = (e-b)/e'$. Set $s = t^r$, and note that s and t are both monomials in $\varphi_1, \ldots, \varphi_{n-1}$. In particular, it suffices to prove that $(sf/\varphi_n^b,\, t\varphi_n^{e'}) = \mathfrak{m}_{\mathcal{Y}_f',z} \subseteq \mathcal{O}_{\mathcal{Y}_f',z}$, which is what we will do. Note that, by Proposition 6.3(iii) and (iv), z lies on the intersection of the v_f' and v_f^*-components of the special fiber when $\lambda' \notin \Gamma_{v_{n-1}}$, and z lies only on the v_f'-component when $\lambda' \in \Gamma_{v_{n-1}}$.

Let

$$f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$$

be the φ_n-adic expansion of f. Let $\eta = s(f - \varphi_n^e)/\varphi_n^b$. By Lemma 6.5(i), $v_f'(f - \varphi_n^e) = v_f(f)$. Since Lemma 6.6 shows that $v_f'(sf/\varphi_n^b) = 0$, we have $v_f'(\eta) = v_f'(s(f-\varphi_n^e)/\varphi_n^b) = v_f(f) - v_f'(f) = e(\lambda_n - \lambda')$. Using Corollary 5.23(i), we conclude that $v_f'(\eta) = 1/(N_n e')$, which generates $\Gamma_{v_f'}$. If $\lambda' \notin \Gamma_{v_{n-1}}$, then Lemma 6.5(iii) shows that $v_f^*(f - \varphi_n^e) = v_f^*(f)$, so $v_f^*(\eta) = v_f^*(sf/\varphi_n^b)$, which equals 0 by Lemma 6.6. Furthermore, since $\eta = s(f-\varphi_n^e)/\varphi_n^b$ and s is a monomial in $\varphi_1, \ldots, \varphi_{n-1}$, Corollary 6.4 and Lemma 6.5(iv) show that the horizontal part of the divisor of η does not contain z.

Since $v_f'(\eta)$ generates $\Gamma_{v_f'}$ and $v_f^*(\eta) = 0$ when $\lambda' \notin \Gamma_{v_{n-1}}$, and the horizontal part of the divisor of η does not intersect z, the divisor of η in $\mathcal{O}_{\mathcal{Y}_{v_f'}^{\mathrm{reg}},z}$ is the prime divisor corresponding to v_f'. By Lemma 6.6(iii), $v_f'(t\varphi_n^{e'}) = 0$. So Spec $\mathcal{O}_{\mathcal{Y}_f',z}/(\eta)$ is a localization of $\mathbb{P}_k^1$ with coordinate given by the reduction of $t\varphi_n^{e'}$ (Lemma 5.1). Since $(t\varphi_n^{e'})^r = s\varphi_n^{re'} = s\varphi_n^{e-b}$ and $\eta = sf/\varphi_n^b - s\varphi_n^{e-b}$, we have

$$\mathcal{O}_{\mathcal{Y}_{v_f'}^{\mathrm{reg}},z}/(sf/\varphi_n^b,\, t\varphi_n^{e'}) = \mathcal{O}_{\mathcal{Y}_{v_f'}^{\mathrm{reg}},z}/(\eta,\, t\varphi_n^{e'}) \cong k.$$

So $\mathfrak{m}_{\mathcal{Y}_{v_f'}^{\mathrm{reg}},z}$ is generated by two elements including sf/φ_n^b, as desired. ☐

Recall that we defined other regular models $\mathcal{Y}_{f,0}'$, $\mathcal{Z}_f$, and $\mathcal{Z}_{f,0}$ of $\mathbb{P}_K^1$ in Definition 6.1.

Corollary 6.10. *In the situation of Theorem 6.9, the divisor D_α is also regular on $\mathcal{Y}_{f,0}'$, $\mathcal{Z}_f$, and $\mathcal{Z}_{f,0}$.*

Proof. This follows because $\mathcal{Y}_{f,0}'$, $\mathcal{Z}_f$, and $\mathcal{Z}_{f,0}$ are all regular blowups of $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ (Remark 6.2). ☐

Corollary 6.11. *Suppose $\nu_K(\alpha) = 2/c$ with $c \geq 3$ odd. If $\mathcal{Y}_w^{\mathrm{reg}}$ is the minimal resolution of the w-model of $\mathbb{P}_K^1$, where $w = [v_0,\, v_1(x) = 2/(c-1)]$, then D_α is regular on $\mathcal{Y}_w^{\mathrm{reg}}$. The special fiber of $\mathcal{Y}_w^{\mathrm{reg}}$ has $(c+1)/2$ irreducible components, one of which is the v_0-component, and the v_0-component meets only the w-component in this model.*

Proof. After a change of variables $z = \pi_K/x$, it suffices to show that the corollary holds if instead $\nu_K(\alpha) = (c-2)/c$ and $w = [v_0,\, v_1(z) = (c-3)/(c-1)]$. But this follows from Theorem 6.9, because in this case, taking f to be the minimal polynomial of α, we have $v_f = [v_0,\, v_1(z) = (c-2)/c]$ and $v'_f = [v_0,\, v_1(z) = (c-3)/(c-1) = ((c-3)/2)/((c-1)/2)]$.

Since $1 > 1/2 > \cdots > 1/((c-1)/2) > 0$ is a 1-path including the shortest 1-path from 1 to $2/(c-1)$, Proposition 5.12 shows that $\mathcal{Y}_w^{\mathrm{reg}}$ includes exactly the valuations $v_\lambda := [v_0, v_1(x) = \lambda]$ for λ in this path. There are $(c+1)/2$ of these, and when $\lambda = 0$, the valuation is v_0. ☐

Remark 6.12. Corollary 6.11 does not immediately follow from Theorem 6.9 without the change of variables. This is because for f the minimal polynomial of α, we would have $v_f = [v_0,\, v_1(x) = 2/c]$, which leads to $v'_f = [v_0,\, v_1(x) = 2/(c+1)]$. The minimal resolution of the v'_f-model of $\mathbb{P}^1_K$ has one more irreducible component on its special fiber than $\mathcal{Y}_w^{\mathrm{reg}}$ in Corollary 6.11 does, since contracting the v'_f-component yields $\mathcal{Y}_w^{\mathrm{reg}}$. We will need the stronger result of Corollary 6.11 to prove the conductor-discriminant inequality for $y^2 = f$ when f is irreducible with roots of valuation $2/c$ for $c \geq 3$ odd.

Remark 6.13. Let α be as in Corollary 6.11, and let f be its minimal polynomial, so that $v_f = [v_0,\, v_1(x) = 2/c]$. Since the shortest 1-path from 1 to $2/c$ is $1 > 1/2 > \ldots > 1/((c-1)/2) > 2/c$ and the shortest 1-path from $2/c$ to 0 is $2/c > 1/((c+1)/2) > 0$, Proposition 5.12 shows that the minimal regular resolution $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ of the v_f-model of $\mathbb{P}^1_K$ includes exactly 2 valuations that are not included in the model $\mathcal{Y}_w^{\mathrm{reg}}$ constructed in the proof of Corollary 6.11 (namely, $v_f = [v_0,\, v_1(x) = 2/c]$ and $[v_0,\, v_1(x) = 1/((c+1)/2)]$). Thus $\mathcal{Z}_{f,0}$ includes exactly one valuation that is not included in $\mathcal{Y}_w^{\mathrm{reg}}$.

Lemma 6.14. *Let $\alpha \in \overline{K}$ such that $\nu_K(\alpha) > 0$, and let f be its minimal monic polynomial. Then D_α is regular on the v_0-model of $\mathbb{P}^1_K$ if and only if f is Eisenstein or linear.*

Proof. The divisor D_α is isomorphic to Spec $\mathcal{O}_K[x]/f \cong \mathcal{O}_K[\alpha]$. This is a DVR if and only if α is a uniformizer of $\mathcal{O}_K(\alpha)$ or if $\mathcal{O}_K[\alpha] = \mathcal{O}_K$; that is, exactly when f is linear or is Eisenstein. ☐

Remark 6.15. The "if" direction of Lemma 6.14 also follows from Theorem 6.9 and Remark 5.31 when f is Eisenstein.

Lemma 6.16. *If f is neither Eisenstein nor linear, then $v'_f \neq v_0$ and $\mathcal{Y}'_{f,0}$ includes $[v_0,\, v_1(x) = 1]$. In particular, $\mathcal{Y}'_{f,0}$ is not the v_0-model of $\mathbb{P}^1_K$.*

Proof. If $v'_f = v_0$, then $\mathcal{Y}_{v'_f}^{\mathrm{reg}} = \mathcal{Y}_{v'_f}$ is the v_0-model and the horizontal part of $\mathrm{div}_0(f)$ is regular on $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ by Theorem 6.9. However, since we assumed that f is neither Eisenstein nor linear, Lemma 6.14 implies that $v'_f \neq v_0$. Since all roots of f have positive valuation, and f is a key polynomial over v_f, we have that $v_f = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$, where φ_n has roots of positive valuation. The same property of φ_n holds for v'_f. By Corollary 5.16(iv), $\mathcal{Y}'_{f,0}$ includes $v := [v_0,\, v_1(x) = 1]$. ☐

6.2. **Counting the components.** Let $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$ be the regular model of $\mathbb{P}^1_K$ from Definition 6.1. In this subsection, we use Proposition 5.12 and Corollary 5.17 to place upper bounds on the number of irreducible components of the special fiber of $\mathcal{Y}_{v'_f}^{\mathrm{reg}}$. The main result of §6.2 is Proposition 6.20. Recall that $e = e(v_n/v_{n-1})$.

Corollary 6.17. *Maintain the notation of Definition 6.1 and Theorem 6.9. Recall, in particular, that $v_f = v_n$ and that $\lambda_0 = \lfloor \lambda_1 \rfloor$. Write*

$$B = \sum_{i=1}^{n} \left(\lfloor \deg(\varphi_i)\lambda_i \rfloor - \deg(\varphi_i)(e(v_{i-1}/v_{i-2})\lambda_{i-1}) + e(v_i/v_{i-1}) \right).$$

and B_0 for the same quantity, but with λ_0 replaced by 0.

(i) *The total number of irreducible components of the special fiber of $\mathcal{Z}_f$ is bounded above by B. The total number of irreducible components of the special fiber of $\mathcal{Z}_{f,0}$ is bounded above by B_0.*

(ii) *If $\lambda' \in \Gamma_{v_{n-1}} = (1/\deg(\varphi_n))\mathbb{Z}$, then the total number of irreducible components of the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ is bounded above by $B - e + 1$. The total number of irreducible components of the special fiber of $\mathcal{Y}'_{f,0}$ is bounded above by $B_0 - e + 1$. Furthermore, if $e = 2$, then $\lambda' \in \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$, so this case applies.*

(iii) *If $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, then the estimates in parts (i) and (ii) can be improved by 1 (i.e., the total number of irreducible components of the special fiber of $\mathcal{Z}_f$ is bounded above by $B - 1$, etc.).*

Proof. Observe that Lemma 4.3(iv) applied to each v_i shows that $\deg(\varphi_i)\lambda_{i-1}$ is an integer for all $1 \leq i \leq n$. So B is an integer. Recall that $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ is the minimal resolution of the v_f-model of $\mathbb{P}^1_K$.

By Corollary 4.4, $N_i = \deg(\varphi_i)$. By Lemma 5.28, $\lambda_n \notin (1/\deg(\varphi_n))\mathbb{Z}$. By summing the expression in Corollary 5.17(ii) for $1 \leq i \leq n$, and adding the component corresponding to $[v_0, v_1(\varphi_1) = \lambda_0]$, the total number of irreducible components on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ is bounded above by $B + 1$. Since $\mathcal{Z}_f$ is obtained by contracting one component of the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$, the number of irreducible components of the special fiber of $\mathcal{Z}_f$ is bounded above by B. The same holds for $\mathcal{Z}_{f,0}$ and B_0, using Corollary 5.17(iv). This proves (i).

If $\lambda' \in (1/\deg(\varphi_n))\mathbb{Z}$, then to count the number of components on the special fiber of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$, we can sum the expression in Corollary 5.17(ii) for $1 \leq i \leq n-1$, add the expression in Corollary 5.17(iii) with λ' substituted for λ_n, and add 1 for $[v_0, v_1(\varphi_1) = \lambda_0]$. This gives at most $B - e + 1$, since $\lambda' < \lambda_n$. This is our upper bound. The same holds for $\mathcal{Y}'_{f,0}$ and B_0, using Corollary 5.17(iv). This proves the first two statements of (ii).

If $e = 2$, then $\deg(\varphi_n)\lambda_n$ is a half-integer by Lemma 5.29, and the shortest 1-path from $\deg(\varphi_n)\lambda_n$ to $\deg(\varphi_n)(\deg(\varphi_n)/\deg(\varphi_{n-1}))\lambda_{n-1}$ begins with $\deg(\varphi_n)\lambda_n > \deg(\varphi_n)\lambda_n - 1/2 \in \mathbb{Z}$. By [OW18, Lemma A.7], $\deg(\varphi_n)\lambda_n - 1/2 = \deg(\varphi_n)\lambda'$. So $\lambda' \in (1/\deg(\varphi_n))\mathbb{Z} = 1/(N_n)\mathbb{Z}$. This finishes the proof of (ii).

In either case above, if $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, Corollary 5.17(ii) shows that our estimate can be improved by 1. This proves (iii). ☐

The following lemma lets us replace B_0 with something simpler.

Lemma 6.18. *Assume $\alpha \notin K$. Let B_0 be as in Corollary 6.17. Then $B_0 \leq \lfloor \deg(\varphi_n)\lambda_n \rfloor + e$, and the inequality is strict unless $n = 1$ or $n = 2$ and φ_2 is Eisenstein.*

Proof. Since $\alpha \notin K$, we have $n \geq 1$, since if $n = 0$, then α would be a root of a key polynomial over v_0, which would be linear by Lemma 4.3(i).

Let B_0' be equal to B_0, except replacing $\lfloor \deg(\varphi_i)\lambda_i \rfloor$ by $\deg(\varphi_i)\lambda_i$ for $1 \leq i \leq n-1$. Then $B_0 \leq B_0'$, and regrouping the terms, we see that

$$B_0' = \lfloor \deg(\varphi_n)\lambda_n \rfloor + e(v_n/v_{n-1}) + \sum_{i=1}^{n-1}(\deg(\varphi_i) - \deg(\varphi_{i+1})e(v_i/v_{i-1}))\lambda_i + e(v_i/v_{i-1})).$$

By Lemma 4.3(iii), $e(v_i/v_{i-1}) = \deg(\varphi_{i+1})/\deg(\varphi_i)$, which means that the summation in the expression for B_0' above is

$$\sum_{i=1}^{n-1}(\deg(\varphi_i)(1 - e(v_i/v_{i-1})^2)\lambda_i + e(v_i/v_{i-1})). \tag{6.19}$$

We need to show that $B_0 - B_0' + (6.19) \leq 0$, with equality holding only when $n = 2$ and $\lambda_1 = 1/e(v_1/v_0)$ or $n = 1$. For $n = 1$ the inequality is immediate, so we assume that $n \geq 2$.

If $i > 1$, then $\deg(\varphi_i)\lambda_i > \deg(\varphi_i)\lambda_{i-1} \geq 1$ by Lemma 4.3(iv). So the ith term in (6.19) is less than $1 - e(v_i/v_{i-1})^2 + e(v_i/v_{i-1})$, which is at most -1 because $e(v_i/v_{i-1}) \geq 2$.

If $i = 1$, then $\deg(\varphi_i) = 1$ by Lemma 4.3(i). Also, $\lambda_1 \geq 1/e(v_1/v_0)$. So the $i = 1$ term of (6.19) is negative unless $\lambda_1 = 1/e(v_1/v_0)$, in which case it is $1/e(v_1/v_0)$. Overall, we see that (6.19) is negative unless $n = 2$ and $\lambda_1 = 1/e(v_1/v_0)$, in which case it is $1/e(v_1/v_0)$. This proves the lemma except in this case.

If $n = 2$ and $\lambda_1 = 1/e(v_1/v_0)$ (which means that φ_2 is Eisenstein by Remark 5.31 and Lemma 4.2), then $B_0' - B_0 = \deg(\varphi_1)\lambda_1 - \lfloor \deg(\varphi_1)\lambda_1 \rfloor = 1/e(v_1/v_0)$. So $B_0 - B_0' + (6.19) = 0$, proving the lemma in this case. ☐

Proposition 6.20. *Maintain the notation of Theorem 6.9. Let $N_{\mathcal{Y}'_{f,0}}$ (resp. $N_{\mathcal{Z}_{f,0}}$) be the number of irreducible components of the special fiber of $\mathcal{Y}'_{f,0}$ (resp. $\mathcal{Z}_{f,0}$). Then*

- $N_{\mathcal{Z}_{f,0}} < \deg(\varphi_n)\lambda_n + e$.
- $N_{\mathcal{Y}'_{f,0}} \leq \deg(\varphi_n)\lambda_n + \frac{1}{2}$ *when* $e = 2$.

If $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, then the estimates can be improved by 1 (i.e., $N_{\mathcal{Z}_{f,0}} < \deg(\varphi_n)\lambda_n + e - 1$), etc.

Proof. The first inequality follows from Corollary 6.17(i) and Lemma 6.18, using the fact that $\deg(\varphi_n)\lambda_n$ is not an integer (Corollary 4.4 and Lemma 5.28). If $e = 2$, we use Corollary 6.17(ii), Lemma 6.18, and the fact that $\deg(\varphi_n)\lambda_n$ is not an integer to get that $N_{\mathcal{Y}'_{f,0}} \leq \deg(\varphi_n)\lambda_n + 1$. By Lemma 5.29, $\deg(\varphi_n)\lambda_n$ is a half-integer, so we have $N_{\mathcal{Y}'_{f,0}} \leq \deg(\varphi_n)\lambda_n + 1/2$.

If $\lambda_1 = 3/c$ in lowest terms with $c \geq 5$, the proof is the same, using Corollary 6.17(iii) and Lemma 6.18. ☐

Note that, when $e = 2$, the bound on $N_{\mathcal{Y}'_{f,0}}$ in Proposition 6.20 is one less than the bound on $N_{\mathcal{Z}_{f,0}}$. This can be seen as a specific consequence of the following.

Lemma 6.21. *Suppose $e = 2$. The only valuation included in $\mathcal{Z}_f$ but not in $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ is v_f''. The same is true for $\mathcal{Z}_{f,0}$ and $\mathcal{Y}'_{f,0}$.*

Proof. It suffices to show that the only valuations included in $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ but not in $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ are v_f and v_f''. By the construction of $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ (Proposition 5.12), the valuations included in $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ but not in $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ are the $v_{n,\lambda}$ in the notation of that proposition. By Lemma 5.19, these valuations correspond exactly to v_f and v''_f. □

We also record the following lemma for use in §9.

Lemma 6.22. *If $e = 2$, then on $\mathcal{Y}'_{f,0}$, the v'_f-component is an even component for both* $\operatorname{div}(f)$ *and* $\operatorname{div}(\pi_K f)$.

Proof. By Lemma 6.5(ii), we have $v'_f(f) = 2\lambda'$. The order of vanishing of f along the component in question is thus $2e(v'_f/v_0)\lambda'$, which is even because $e(v'_f/v_0)\lambda'$ is an integer by definition. The order of vanishing of π_K is $e(v_f/v_0) = \deg(f) = 2\deg(\varphi_n)$ by Lemma 4.3(iii) and Corollary 4.4, and this is again even. □

7. Discriminant bonus calculations

Similarly to §6, we fix α, this time in the maximal ideal of $\mathcal{O}_{K^{\mathrm{sep}}}$, and we let $f(x)$ be the minimal polynomial of α. We assume $f(x)$ has degree at least 2. The main goal of this section is to compute a lower bound on the *discriminant bonus* $\operatorname{db}_K(\alpha)$ of α in terms of the invariants appearing in the unique Mac Lane valuation v_f over which f is a proper key polynomial (Proposition 4.9(iv)). Here, we write

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_n(\varphi_n) = \lambda_n].$$

Recall from Definition 2.4 that $\operatorname{db}_K(\alpha)$ is defined to be $\Delta_{f,K} - \Delta_{K(\alpha)/K}$. Recall also that every algebraic (resp. separable) extension L/K is assumed to be embedded into $\overline{K}$ (resp. K^{sep}) via the inclusion ι_L.

The main results of §7 are Corollary 7.7 and Proposition 7.31. Combined, these results roughly say that $\operatorname{db}_K(\alpha) \geq 2\deg(\varphi_n)\lambda_n$ (give or take), and they can be used as black boxes if the reader would like to glance at the results and then skip to §8.

Our results are examples of the following principle:

Principle 7.1. The better α can be approximated by an element of lower K-degree (which corresponds to λ_n being large), the greater $\operatorname{db}_K(\alpha)$ is.

The element of lower K-degree that we use is a root β of φ_n, and λ_n being large means that β approximates α well in the sense that $\nu_K(\varphi_n(\alpha)) = \lambda_n$ (Corollary 4.12).

For example, if $n = 1$ and $\varphi_1(x) = x$, then $\beta = 0$ and $\nu_K(\alpha) = \lambda_1$. It is easy to see that $\operatorname{db}_K(\alpha)$ increases as $\nu_K(\alpha) = \lambda_1$ does: this is the content of Lemma 7.3(ii). However, it is a bit less straightforward to make Principle 7.1 concrete when $\beta \notin K$, and much less straightforward to do so when $\beta \notin K(\alpha)$. Essentially, bounding $\operatorname{db}_K(\alpha)$ from below relies on having good lower bounds on $\nu_K(\sigma(\alpha) - \alpha) - \nu_K(\sigma(\pi_{K(\alpha)}) - \pi_{K(\alpha)})$, where σ ranges over K-embeddings of $K(\alpha)$ into $\overline{K}$ and $\pi_{K(\alpha)}$ is a uniformizer of $K(\alpha)$ (Lemma 7.2). If $\nu_K(\alpha)$ is large, then such a bound is proven straightforwardly in Lemma 7.3. If $\beta \in K(\alpha)$ of lower degree than α is close to α, then at least for σ that fix β, one can replace α with $\alpha - \beta$ when computing $\nu_K(\sigma(\alpha) - \alpha) - \nu_K(\sigma(\pi_{K(\alpha)}) - \pi_{K(\alpha)})$, and thus treat α as if it has large valuation. But if $\beta \notin K(\alpha)$, this doesn't work, because $K(\alpha - \beta) \neq K(\alpha)$. The length of §7 is in large part due to the difficulty of circumventing this issue.

After some preliminary lemmas in §7.1, we dispense with the case $n = 1$ in §7.2. In §7.3, we prove the inequality when every K-conjugate of β is equidistant from every K-conjugate of α. If $\deg(f)$ is a power of $\operatorname{char} k$, this actually means that $K(\alpha)$ and $K(\beta)$ are linearly disjoint over K — as mentioned above, this is the most difficult case to get a good lower bound on the discriminant bonus. The somewhat odd-looking Proposition 7.18 splits this up into two subcases, which are proven separately.

In §7.4, we prove the inequality in general by letting M/K be an extension over which the M-conjugates of α are all equidistant from the M-conjugates of β. We show that we can take M to lie inside the intersection of $K(\alpha)$ and $K(\beta)$. The general proof then proceeds by using the equidistant case to prove an analogous inequality over M (Lemma 7.30), and then descending the result to K.

Notation for §7. As running notation throughout §7, we write

- $\lambda_i = b_i/c_i$ in lowest terms.
- $e := \deg(f)/\deg(\varphi_n)$.

7.1. **Preliminaries on the discriminant bonus.** The following lemma gives a convenient way of calculating the discriminant bonus.

Lemma 7.2. *Let $\alpha \in K^{\text{sep}}$ and let $L = K(\alpha)$ with uniformizer π_L. Then*

$$\mathrm{db}_K(\alpha) = \sum_{\substack{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \\ \sigma(\alpha) \neq \alpha}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)).$$

Proof. Let f be the minimal polynomial of α and g the minimal polynomial of the uniformizer π_L of L/K. Since k is algebraically closed, by [Ser79, Chapter 3, Lemma 3], we have $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$, and $\Delta_{L/K} = \Delta_{g,K}$. By [Ser79, Chapter 3, Corollary 2], we have $\Delta_{f,K} = \nu_L(f'(\alpha))$ and $\Delta_{g,K} = \nu_L(g'(\pi_L))$. Now

$$\begin{aligned}
\Delta_{f,K} &= \sum_{\sigma \neq \tau \in \mathrm{Hom}_K(L, \overline{K})} \nu_K(\sigma(\alpha) - \tau(\alpha)) \\
&= [L : K] \sum_{\substack{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \\ \sigma(\alpha) \neq \alpha}} \nu_K(\sigma(\alpha) - \alpha) \\
&= \sum_{\substack{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \\ \sigma(\alpha) \neq \alpha}} \nu_L(\sigma(\alpha) - \alpha)
\end{aligned}$$

The lemma follows by expressing $\Delta_{g,K}$ as $\Delta_{f,K}$ was expressed above. □

Lemma 7.3 will be used repeatedly throughout this section. It is a loose generalization of [Ser79, IV, Lemma 1] to non-Galois extensions. Its consequence, Corollary 7.4, is the most fundamental example of Principle 7.1.

Lemma 7.3. *Let L/K be a finite separable extension and let $\sigma \in \mathrm{Hom}_K(L, K^{\text{sep}}) \setminus \{\iota_L\}$.*

(i) *The quantity $\nu_L(\sigma(\pi_L) - \pi_L)$ does not depend on the uniformizer π_L of L chosen.*

(ii) *If $\gamma \in L$ such that there exists $\eta \in K$ with $\nu_L(\gamma - \eta) = a$, then $\nu_L(\sigma(\gamma) - \gamma) \geq \nu_L(\sigma(\pi_L) - \pi_L) + a - 1$.*

Proof. Let z in the maximal ideal $\mathfrak{m}_L$ of $\mathcal{O}_L$ be such that $\nu_0 := \nu_L(\sigma(z) - z)$ is minimal. One checks easily that z is a uniformizer of L. Let π_L be another uniformizer of L. Then

$\pi_L = uz$ for some $u \in \mathcal{O}_L^\times$, and $\sigma(\pi_L) - \pi_L = \sigma(uz) - uz = \sigma(u)(\sigma(z) - z) + z(\sigma(u) - u)$. Since the residue field of L is the same as that of K, there exists $b \in \mathfrak{m}_L$ such that $u = b + \eta$ with $\eta \in K$. So $\sigma(u) - u = \sigma(b) - b$. Thus

$$\nu_L(z(\sigma(u) - u)) = \nu_L(z(\sigma(b) - b)) \geq \nu_L(z(\sigma(z) - z)) > \nu_L(\sigma(z) - z) = \nu_0.$$

It follows that $\nu_L(\sigma(\pi_L) - \pi_L) = \nu_0$, proving (i).

For part (ii), we may assume $\eta = 0$. Write $\gamma = u\pi_L^a$. Then $\sigma(\gamma) - \gamma = \sigma(u)(\sigma(\pi_L^a) - \pi_L^a) + \pi_L^a(\sigma(u) - u)$. The second term has valuation at least $\nu_0 + a$, as can be seen by replacing u by b as in the proof of part (i). For the first term, if we write $\sigma(\pi_L) = \pi_L + \epsilon$, then $\nu_0 = \nu_L(\epsilon) \geq \nu_L(\pi_L)$ because $\epsilon \in \mathfrak{m}_L$. So $\sigma(\pi_L^a) - \pi_L^a = (\pi_L + \epsilon)^a - \pi_L^a$, and ν_L of this is at least $\nu_0 + (a-1)\nu_L(\pi_L)$, proving (ii). □

Corollary 7.4. *If $[K(\alpha) : K] = d$ and $\nu_L(\alpha) = a$, then $\mathrm{db}_K(\alpha) \geq (d-1)(a-1)$.*

Proof. This follows from Lemmas 7.3 and 7.2, taking $L = K(\alpha)$, $\gamma = \alpha$, and $\eta = 0$ in Lemma 7.3 (ii). □

We also state the following variation of [Ser79, IV, Proposition 3] for extensions that are not necessarily Galois.

Lemma 7.5. *Let $K''/K'/K$ be a tower of finite separable field extensions with uniformizers $\pi_{K''}$, $\pi_{K'}$, and π_K, respectively. Let $\sigma \in \mathrm{Hom}_K(K', K^{\mathrm{sep}}) \setminus \{\iota_{K'}\}$ and write $S = \{s \in \mathrm{Hom}_K(K'', K^{\mathrm{sep}}) \mid s|_{K'} = \sigma\}$. Then*

$$\nu_K(\sigma(\pi_{K'}) - \pi_{K'}) = \sum_{s \in S} \nu_K(s(\pi_{K''}) - \pi_{K''}).$$

Proof. It suffices to show that $a := \sigma(\pi_{K'}) - \pi_{K'}$ and $b := \prod_{s \in S}(\pi_{K''} - s(\pi_{K''}))$ generate the same ideal in $\mathcal{O}_{K''}$. Let $j(X)$ be the minimal polynomial of $\pi_{K''}$ over K'. Since the roots of $j(X)$ are all the K'-conjugates of $\pi_{K''}$, it follows that $\sigma j(X) = \prod_{s \in S}(X - s(\pi_{K''}))$.

The proof is now exactly the same as the proof of [Ser79, IV, Proposition 3], but we include it here for clarity. To show $a \mid b$, note that since the coefficients of $j(X)$ all lie in the maximal ideal of $\mathcal{O}_{K'}$, we have that a divides all coefficients of $\sigma j(X) - j(X)$. So a divides $\sigma j(\pi_{K''}) - j(\pi_{K''}) = \sigma j(\pi_{K''}) = b$.

To show that b divides a, write $\pi_{K'}$ as a polynomial $g(\pi_{K''})$ over K. Then $g(X) - \pi_{K'} \in K'[X]$ has $\pi_{K''}$ as a root, so it is divisible by $j(X)$ in $\mathcal{O}_{K'}[X]$. In particular, $g(X) - \pi_{K'} = j(X)h(X)$ for some $h(X) \in \mathcal{O}_{K'}[X]$. Applying σ to both sides of this equation and plugging in $\pi_{K''}$ for X yields

$$\pi_{K'} - \sigma(\pi_{K'}) = \sigma j(\pi_{K''})\sigma h(\pi_{K''}).$$

Since the left hand side equals a and $\sigma j(\pi_{K''}) = b$, we have $b \mid a$. □

7.2. **Inductive valuation length 1.** Assume for §7.2 that the inductive length n of the valuation v_f is 1. Write $\lambda_1 = b_1/c_1$ in lowest terms. By Lemma 4.3(iii), $\deg(f) = e = c_1$. Note that $c_1 \geq 2$ since $\deg(f) \geq 2$.

Proposition 7.6. *The discriminant bonus $\mathrm{db}_K(\alpha)$ is at least $(b_1 - 1)(c_1 - 1)$.*

Proof. Since $\varphi_1 = x - a$ for some $a \in K$ by Lemma 4.3(i), and replacing α by $\alpha - a$ does not change the discriminant bonus, we may make a change of variables over K and assume $\varphi_1 = x$. By Corollary 4.12, $\nu_K(\alpha) = \lambda_1 = b_1/c_1$. Let $L = K(\alpha)$. Then $\nu_L(\alpha) = b_1$, and the result follows from Corollary 7.4. □

The next corollary may look strangely phrased, but it is written so as to be parallel to Proposition 7.31.

Corollary 7.7. *Write $\lambda_1 = b_1/c_1$ in lowest terms. If $c_1 = 2$, then $\mathrm{db}_K(\alpha) \geq 2\deg(\varphi_1)\lambda_1 - 1$. If $c_1 \geq 3$ and $b_1 \geq 3$, then $\mathrm{db}_K(\alpha) \geq 2(\lfloor \deg(\varphi_1)\lambda_1 \rfloor + e - 1)$. Equality cannot hold in this case unless $b_1 = 3$ or $\lambda_1 = 4/3$.*

Proof. Recall that $\deg(f) = e = c_1$, by Lemma 4.3(iii). If $c_1 = 2$, then $2\deg(\varphi_1)\lambda_1 - 1 = b_1 - 1 = (b_1 - 1)(c_1 - 1)$, and the corollary follows from Proposition 7.6.

If $\deg(f) = c_1 \geq 3$ and $b_1 \geq 3$, then it suffices to prove $(b_1-1)(c_1-1) \geq 2(\lfloor b_1/c_1 \rfloor + c_1 - 1)$, or in other words that $(b_1 - 3)(c_1 - 1) - 2\lfloor b_1/c_1 \rfloor \geq 0$. Since the left-hand side of this expression is increasing in c_1 and b_1/c_1 is in lowest terms, one checks that the left-hand side is non-negative, and is equal to zero only when $b_1 = 3$ or $\lambda_1 = 4/3$. □

7.3. **Inductive valuation length ≥ 2: the equidistant case.** Assume for §7.3 that the inductive length n of the valuation v_f is at least 2. Fix a root $\beta \in \overline{K}$ of φ_n. By [Rüt14, Lemma 4.33], we can make a small adjustment to φ_n without changing the valuation v_f. Thus we may and do assume that φ_n is a *separable* polynomial. The other key assumption for §7.3 is the following.

Assumption 7.8. The quantity $\nu_K(\beta - \alpha)$ does not change when β is replaced by any of its K-conjugates. In other words, all roots of φ_n are equidistant from α.

Lemma 7.9.

(i) *We have $\lambda_n = \nu_K(\alpha - \beta)\deg(\varphi_n)$.*
(ii) *If α' and α'' are any K-conjugates of α, then $\nu_K(\alpha'' - \alpha') \geq \nu_K(\alpha - \beta)$.*

Proof. By Corollary 4.12, $\lambda_n = \nu_K(\varphi_n(\alpha))$, which is equal to $\sum \nu_K(\alpha - \beta')$, where β' ranges over the K-conjugates of β. By Assumption 7.8, each term in the sum is equal, and part (i) follows. For part (ii), we may assume $\alpha'' = \alpha$. If $\nu_K(\alpha - \alpha') < \nu_K(\alpha - \beta)$, then $\nu_K(\alpha' - \beta) < \nu_K(\alpha - \beta)$. There is a K-automorphism of $\overline{K}$ taking α' to α and β to some conjugate β'. Since all K-automorphisms of $\overline{K}$ are isometries, we obtain $\nu_K(\alpha - \beta') = \nu_K(\alpha' - \beta) < \nu_K(\alpha - \beta)$, which contradicts Assumption 7.8. This proves (ii). □

If $\deg(f)$ is not a power of the characteristic k, we can directly get a lower bound on the discriminant bonus in terms of λ_n.

Proposition 7.10. *Assume $\deg(f)$ is not a power of $\mathrm{char}\, k$.*

(i) *If $e = 2$, then $\mathrm{db}_K(\alpha) \geq 2\deg(\varphi_n)\lambda_n - 1$. Equality holds only if $n = 2$, φ_2 is Eisenstein, and $\deg(\varphi_2)$ is a power of $\mathrm{char}\, k$.*
(ii) *If $e > 2$, then $\mathrm{db}_K(\alpha) > 2\deg(\varphi_n)\lambda_n + 2e$.*

Proof. Let r be the prime-to-$\mathrm{char}\, k$ part of $\deg(f)$ (so $r = \deg(f)$ if $\mathrm{char}\, k = 0$). By assumption, $r \geq 2$. Write $L = K(\alpha)$. If M is the intersection of L with the maximal tamely ramified extension of K, then L/M has $\mathrm{char}\, k$-power degree, which means that M/K has degree r. Since k is algebraically closed, M/K is automatically Galois.

There are $\deg(f)(r-1)/r$ K-embeddings of L into $\overline{K}$ that do not restrict to ι_M. If σ is one of these, then $\nu_L(\sigma(\pi_M) - \pi_M) = [L : M]\nu_M(\sigma(\pi_M) - \pi_M) = [L : M]$, since M/K

is tamely ramified. By taking $\gamma = \pi_M$ and $\eta = 0$ in Lemma 7.3(ii), we conclude that $\nu_L(\sigma(\pi_L) - \pi_L) = 1$. By Lemma 7.9,

$$\nu_L(\sigma(\alpha) - \alpha) \geq [L : K]\lambda_n / \deg(\varphi_n) = e\lambda_n. \tag{7.11}$$

On the other hand, since φ_1 is linear, $\nu_L(\sigma(\alpha) - \alpha) = \nu_L(\sigma(\varphi_1(\alpha)) - \varphi_1(\alpha))$ for any K-embedding σ of L into $\overline{K}$. So even for the remaining $\deg(f)/r - 1$ embeddings σ (not including ι_L) for which $\sigma|_M$ is trivial, we still know from Lemma 7.3(ii) that $\nu_L(\sigma(\alpha)-\alpha) - \nu_L(\sigma(\pi_L) - \pi_L) = \nu_L(\sigma(\varphi_1(\alpha)) - \varphi_1(\alpha)) - \nu_L(\sigma(\pi_L) - \sigma(\pi_L)) \geq \nu_L(\varphi_1(\alpha)) - 1$. By Corollary 4.12, $\nu_L(\varphi_1(\alpha)) = \deg(f)\lambda_1$. So the contribution to $\mathrm{db}_K(\alpha)$ from each of these terms is at least $\deg(f)\lambda_1 - 1$. Combining this with (7.11) and Lemma 7.2, we see that

$$\mathrm{db}_K(\alpha) \geq \frac{r-1}{r}\deg(f)(e\lambda_n - 1) + \left(\frac{\deg(f)}{r} - 1\right)(\deg(f)\lambda_1 - 1). \tag{7.12}$$

Note that $e \geq 2$ and $r \geq 2$. By Lemma 4.3(iv), (vi) applied to φ_2 and v_1, we have $\deg(\varphi_2)\lambda_1 \geq 1$, which means that $\deg(f)\lambda_1 \geq e$, with equality only when $n = 2$, and $\lambda_1 = 1/\deg(\varphi_2)$ (i.e., φ_2 is Eisenstein). Also, $\lambda_n \geq \lambda_2 > 1$ by Lemma 4.3(v). We deduce the following string of (in)equalities, starting with (7.12):

$$\begin{aligned}
\mathrm{db}_K(\alpha) &\geq \frac{r-1}{r}\deg(f)(e\lambda_n - 1) + \left(\frac{\deg(f)}{r} - 1\right)(\deg(f)\lambda_1 - 1) \\
&\geq \frac{r-1}{r}\deg(f)(e\lambda_n - 1) + \left(\frac{\deg(f)}{r} - 1\right)(e - 1) \\
&\overset{(1)}{\geq} \frac{1}{2}\deg(f)(e\lambda_n + e - 2) - e + 1 \\
&= \frac{e^2}{2}\deg(\varphi_n)\lambda_n + \frac{e(e-2)}{2}\deg(\varphi_n) - e + 1 \\
&= 2\deg(\varphi_n)\lambda_n + \frac{e^2-4}{2}\deg(\varphi_n)\lambda_n + \frac{e(e-2)}{2}\deg(\varphi_n) - e + 1, \qquad (7.13)
\end{aligned}$$

where inequality (1) follows because $\lambda_n > 1$, so setting $r = 2$ minimizes the quantity in the line above. If $e = 2$, this proves (i) (and furthermore, equality can hold only when $r = n = 2$ and φ_2 is Eisenstein). Now assume $e \geq 3$. Since $\deg(\varphi_n) \geq 2$ and $\lambda_n > 1$, we have that (7.13) is greater than $2\deg(\varphi_n)\lambda_n + 2e^2 - 3e - 3$. This is at least $2\deg(\varphi_n)\lambda_n + 2e$ for $e \geq 3$, proving part (ii). □

Now, we examine the case where $\deg(f)$ is a prime power.

Proposition 7.14. *Suppose* $\deg(f)$ *is a prime power.*

- (i) $K(\alpha)$ *and* $K(\beta)$ *are linearly disjoint over* K.
- (ii) $\deg(f)$ *is a power of the characteristic of* k.

Proof. By Corollary 4.4, $\deg(f) = \mathrm{lcm}(c_1, \ldots, c_n)$, and this is a strict multiple of $\deg(\varphi_n) = \mathrm{lcm}(c_1, \ldots, c_{n-1})$. If $\deg(f)$ is a prime power, the only way this can happen is if $c_n = \deg(f)$. By Lemma 7.9(i), $\nu_K(\alpha - \beta) = \lambda_n/\deg(\varphi_n)$, which has denominator $c_n \deg(\varphi_n)$. Thus

$$[K(\alpha, \beta) : K] \geq c_n \deg(\varphi_n) = \deg(f)\deg(\varphi_n) = \deg(\alpha)\deg(\beta).$$

The inequality above is thus an equality, proving (i). Since there is only one tamely ramified extension of K of any given degree, the only way that two nontrivial prime power degree extensions of K can be linearly disjoint is if they are wild. This proves (ii). □

Lemma 7.15. *If $g \in K[x]$ is a polynomial with $\deg(g) < \varphi_{i+1}$ for some $1 \le i < n$, then $v_f(g) - v_f(g') \le \max_{j \le i}(\lambda_j - v_f(\varphi_j')) = \max_{j \le i}(v_f(\varphi_j) - v_f(\varphi_j'))$.*

Proof. By taking the φ_i-adic expansion of g, then taking the φ_{i-1}-adic expansion of the coefficients in the φ_i-adic expansion, then taking the φ_{i-2}-adic expansions of all of the coefficients in the φ_{i-1}-adic expansions, and so on down to the φ_1-adic expansions, we can write g as a sum of monomials in the φ_j, $j \le i$, with coefficients in K. Furthermore, by the definition of inductive valuations, $v_f(g)$ is equal to the minimum value of v_f on one of these monomials. By the product rule, differentiating a monomial in $\varphi_1, \ldots, \varphi_i$ decreases its valuation by at most $\max_{j \le i}(v_f(\varphi_j) - v_f(\varphi_j')) = \max_{j \le i}(\lambda_j - v_f(\varphi_j'))$. Thus this quantity also bounds $v_f(g) - v_f(g')$ from above. □

Lemma 7.16. *Suppose all c_i (that is, all denominators of λ_i) are powers of $p = \operatorname{char} k$. Let $M = \max_i(\lambda_i - v_f(\varphi_i')) = \max_i(v_f(\varphi_i) - v_f(\varphi_i'))$. If $\lambda_j - v_f(\varphi_j') = M$, then $j \in \{n-1, n\}$.*

Proof. Let j be minimal such that $\lambda_j - v_f(\varphi_j') = M$, and suppose for a contradiction that $j < n - 1$. Consider the φ_{j+1}-adic expansion of φ_{j+2}:

$$\varphi_{j+2} = \varphi_{j+1}^s + a_{s-1}\varphi_{j+1}^{s-1} + \cdots + a_0. \tag{7.17}$$

Since φ_{j+2} is a key polynomial over v_{j+1}, we have $s\lambda_{j+1} = v_f(a_0) = v_{j+1}(a_0)$ (Lemma 4.3(ii)). Now, $s = \operatorname{lcm}(c_1, \ldots, c_{j+1})/\operatorname{lcm}(c_1, \ldots, c_j) = c_{j+1}/c_j$, since the c_j are all pth powers. So $v_{j+1}(a_0) = v_f(a_0)$ has denominator equal to c_j. In particular, if

$$a_0 = d_{c_j/c_{j-1}-1}\varphi_j^{c_j/c_{j-1}-1} + \cdots + d_0,$$

is the φ_j-adic expansion of a_0, the fact that the c_i are increasing and that $v_f(d_i) \in (1/c_{j-1})\mathbb{Z}$ for all i shows that no two terms of the expansion have identical valuations under v_f. This means that there is a unique term $S = d_\ell\varphi_j^\ell$ such that $v_f(S) = v_f(a_0)$. We have $p \nmid \ell$ because the denominator of $v_f(a_0)$ is c_j. Since $S' = S(d_\ell'/d_\ell + \ell\varphi_j'/\varphi_j)$, and $v_f(d_\ell) - v_f(d_\ell') < M$ by Lemma 7.15 and the fact that $\deg(d_\ell) < \deg(\varphi_j)$ (this is where we use the minimality of j), we have that $v_f(S) - v_f(S') = M$. Since all other terms T in the φ_j-adic expansion of a_0 satisfy $v_f(S) < v_f(T)$, applying Lemma 7.15 again yields $v_f(T') \ge v_f(T) - M > v_f(S) - M = v_f(S')$. So $v_f(a_0) - v_f(a_0') = v_f(S) - v_f(S') = M$.

By Lemma 7.15, all monomials T in (7.17) other than φ_{j+1}^s satisfy $v_f(T) - v_f(T') \le M$. Now, $\deg(\varphi_{j+2})$, $\deg(\varphi_{j+1})$, and s are all powers of p. Thus $v_f(s) > 0$ and consequently $v_f(\varphi_{j+1}^s) - v_f((\varphi_{j+1}^s)') < M$. Since a_0 and φ_{j+1}^s are the dominant terms of (7.17), and $v_f(a_0) - v_f(a_0') = M$ is maximal among all the $v_f(T) - v_f(T')$, we have $v_{j+1}(\varphi_{j+2}) - v_{j+1}(\varphi_{j+2}') = M$. But $\lambda_{j+2} = v_f(\varphi_{j+2}) > v_{j+1}(\varphi_{j+2})$, so $\lambda_{j+2} - v_f(\varphi_{j+2}') > M$, contradicting the definition of M. This proves the lemma. □

Proposition 7.18. *Suppose $\deg(f)$ is a prime power, and recall that $e = \deg(f)/\deg(\varphi_n)$. Then $v_f(\varphi_n') \le \lambda_n - \lambda_1$ or $v_f(f') \le e\lambda_n - \lambda_1$.*

Proof. By Proposition 7.14(ii), $\deg(f)$ is a power of $\operatorname{char} k = p$. By Corollary 4.4, the hypothesis of Lemma 7.16 holds for v_f. Consider valuations $v_\lambda := [v_f, v_{n+1}(f) = \lambda]$, where λ is a decreasing sequence of rational numbers converging to $v_f(f) = e\lambda_n$, all of whose denominators are powers of p. The hypothesis of Lemma 7.16 holds for v_λ as well, so define M_λ analogously to M in Lemma 7.16, but using v_λ instead of v_f. That is,

$$M_\lambda := \max(\max_{1\le i\le n}(v_\lambda(\varphi_i) - v_\lambda(\varphi_i')), v_\lambda(f) - v_\lambda(f')) = \max(\max_{1\le i\le n}(\lambda_i - v_f(\varphi_i')), \lambda - v_f(f')),$$

where the second equality holds because v_f agrees with v_λ on all polynomials with degree less than $\deg(f)$. By Lemma 4.3(i), $\varphi_1' = 1$. So $\lambda_1 - v_f(\varphi_1') = \lambda_1$, which implies $M_\lambda \ge \lambda_1$.

Applying Lemma 7.16 to each v_λ shows that for each λ in the sequence, either $\lambda - v_f(f') = M_\lambda \ge \lambda_1$ or $\lambda_n - v_f(\varphi_n') = M_\lambda \ge \lambda_1$. Letting $\lambda \to e\lambda_n$, we conclude that either $e\lambda_n - v_f(f') \ge \lambda_1$ or $\lambda_n - v_f(\varphi_n') \ge \lambda_1$. This finishes the proof. □

In Propositions 7.19 and 7.24 below, we prove that $\mathrm{db}_K(\alpha) > 2\deg(\varphi_n)\lambda_n + 2e$ in each of the two cases in Proposition 7.18. For the rest of §7.3, let $L = K(\alpha)$, $K' = K(\beta)$, and $L' = LK' = K(\alpha, \beta)$, with $\deg(f)$ a prime power. Furthermore, fix uniformizers π_L, $\pi_{K'}$, and $\pi_{L'}$ of K', L, and L', respectively. By Proposition 7.14(i), L and K' are linearly disjoint over K.

Proposition 7.19. *Suppose* $\deg(f)$ *is a prime power and* $v_f(\varphi_n') \le \lambda_n - \lambda_1$. *Then* $\mathrm{db}_K(\alpha) > 2\deg(\varphi_n)\lambda_n + 2e$.

Proof. Since L and K' are linearly disjoint over K, we have $\deg(\varphi_n)\Delta_{f,K} = \Delta_{f,K'}$. Recall that if $M''/M'/M$ are finite separable totally ramified field extensions, then

$$\Delta_{M''/M} = [M' : M]\Delta(M''/M') + \Delta(M'/M).$$

Applying this to $L'/L/K$ and then to $L'/K'/K$, we compute

$$\begin{aligned} \mathrm{db}_K(\alpha) &= \Delta_{f,K} - \Delta_{L/K} \\ &= \frac{1}{\deg(\varphi_n)}(\Delta_{f,K'} - [K' : K]\Delta_{L/K}) \\ &> \frac{1}{\deg(\varphi_n)}(\Delta_{f,K'} - \Delta_{L'/K}) \\ &= \frac{1}{\deg(\varphi_n)}(\Delta_{f,K'} - \Delta_{L'/K'} - [L' : K']\Delta_{K'/K}) \\ &= \frac{1}{\deg(\varphi_n)}\mathrm{db}_{K'}(\alpha) - e\Delta_{K'/K}. \end{aligned} \tag{7.20}$$

Since $\beta \in K'$, Lemma 7.3(ii) shows that $\nu_{L'}(\sigma(\alpha) - \alpha) - \nu_{L'}(\sigma(\pi_{L'}) - \pi_{L'}) \ge \nu_{L'}(\alpha - \beta) - 1$. Since $[L' : K] = \deg(f)\deg(\varphi_n)$, Lemma 7.9(i) implies that this is in turn equal to $\deg(f)\lambda_n - 1$. This means that $\mathrm{db}_{K'}(\alpha) \ge (\deg(f) - 1)(\deg(f)\lambda_n - 1)$. On the other hand, $\Delta_{K'/K} \le \nu_{K'}(\varphi_n'(\beta)) = \deg(\varphi_n)(\nu_K(\varphi_n'(\beta))) = \deg(\varphi_n)(v_f(\varphi_n')) < \deg(\varphi_n)\lambda_n$, where the last equality comes from Proposition 4.11 applied to φ_n' and β and the inequality is by our assumption. Plugging this all into (7.20), and noting that e and $\deg(\varphi_n)$ are at least 3 and

$\lambda_n > 1$ by Lemma 4.3(v) gives

$$\begin{aligned}
\mathrm{db}_K(\alpha) &> \frac{1}{\deg(\varphi_n)}(\deg(f)-1)(\deg(f)\lambda_n - 1) - e\deg(\varphi_n)\lambda_n \\
&= e(\deg(f) - \deg(\varphi_n) - 1 - 1/\lambda_n)\lambda_n + \frac{1}{\deg(\varphi_n)} \\
&> e((e-1)\deg(\varphi_n) - 2)\lambda_n \\
&= e((e-2)\deg(\varphi_n) - 2)\lambda_n + e\deg(\varphi_n)\lambda_n \\
&> e((e-2)\deg(\varphi_n) - 2)\lambda_n + 2e + e\lambda_n \\
&= 2\deg(\varphi_n)\lambda_n + 2e + ((e^2 - 2e - 2)\deg(\varphi_n) - e)\lambda_n \\
&\geq 2\deg(\varphi_n)\lambda_n + 2e + (3(e^2 - 2e - 2) - e)\lambda_n \\
&\geq 2\deg(\varphi_n)\lambda_n + 2e.
\end{aligned}$$

The proposition is proved. ☐

For Lemma 7.22 below, note that, if L and K' are linearly disjoint over K, there is a canonical bijection

$$\mathrm{Hom}_K(L, \overline{K}) \times \mathrm{Hom}_K(K', \overline{K}) \to \mathrm{Hom}_K(L', \overline{K}). \tag{7.21}$$

If $\sigma \in \mathrm{Hom}_K(L, \overline{K})$ and $\tau \in \mathrm{Hom}_K(K', \overline{K})$, we denote by $\sigma\tau$ the resulting K-embedding of L' into $\overline{K}$. Furthermore, recall that α and β are taken to live inside $\overline{K}$, so there are distinguished inclusion elements $\iota_L \in \mathrm{Hom}_K(L, \overline{K})$ and $\iota_{K'} \in \mathrm{Hom}_K(K', \overline{K})$.

Lemma 7.22. *Suppose* $\deg(f)$ *is a prime power. Fix* $\tau \in \mathrm{Hom}_K(K', \overline{K})$. *There exists* $\sigma_\tau \in \mathrm{Hom}_K(L, \overline{K})$ *such that for all* $\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus \{\sigma_\tau\}$,

$$\nu_{L'}(\sigma\tau(\pi_{L'}) - \pi_{L'}) \leq \nu_{L'}(\sigma(\alpha) - \sigma_\tau(\alpha)) - \deg(f)\lambda_n + 1. \tag{7.23}$$

If $\tau = \iota_{K'}$, *then we can take* $\sigma_\tau = \iota_L$.

Proof. By Proposition 7.14(i), L and K' are linearly disjoint over K. By Lemma 7.9(i), along with the fact that $[L' : K] = \deg(f)\deg(\varphi_n)$, we have $\nu_{L'}(\alpha - \beta) = \deg(f)\lambda_n$. Combining this with Lemma 7.3(ii), we have

$$\begin{aligned}
\nu_{L'}(\sigma\tau(\pi_{L'}) - \pi_{L'}) &\leq \nu_{L'}(\sigma\tau(\alpha-\beta) - (\alpha - \beta)) - \nu_{L'}(\alpha - \beta) + 1 \\
&= \nu_{L'}(\sigma\tau(\alpha-\beta) - (\alpha - \beta)) - \deg(f)\lambda_n + 1.
\end{aligned}$$

So it suffices to find σ_τ such that $\nu_{L'}(\sigma\tau(\alpha - \beta) - (\alpha - \beta)) \leq \nu_{L'}(\sigma(\alpha) - \sigma_\tau(\alpha))$ for all $\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus \{\sigma_\tau\}$.

Now, $\sigma\tau(\alpha - \beta) - (\alpha - \beta) = \sigma(\alpha) - \alpha - (\tau(\beta) - \beta)$. If $\tau = \iota_{K'}$, then taking $\sigma_\tau = \iota_L$ clearly works. For other τ, take σ_τ to be any element of $\mathrm{Hom}_K(L, \overline{K})$ such that $\nu_{L'}(\sigma_\tau(\alpha) - \alpha - (\tau(\beta) - \beta))$ is maximal. Then

$$\sigma(\alpha) - \sigma_\tau(\alpha) = (\sigma(\alpha) - \alpha - (\tau(\beta) - \beta)) - (\sigma_\tau(\alpha) - \alpha - (\tau(\beta) - \beta)).$$

By assumption,

$$\nu_{L'}(\sigma(\alpha) - \alpha - (\tau(\beta) - \beta)) \leq \nu_{L'}(\sigma_\tau(\alpha) - \alpha - (\tau(\beta) - \beta)).$$

This means that

$$\nu_{L'}(\sigma(\alpha) - \alpha - (\tau(\beta) - \beta)) \leq \nu_{L'}(\sigma(\alpha) - \sigma_\tau(\alpha)),$$

and we are done.

$\square$

Proposition 7.24. *Suppose* $\deg(f)$ *is a prime power and* $v_f(f') \leq e\lambda_n - \lambda_1$*. Then* $\mathrm{db}_K(\alpha) > 2\deg(\varphi_n)\lambda_n + 2e$*.*

Proof. For each $\tau \in \mathrm{Hom}_K(K', \overline{K})$, Lemma 7.22 allows us to choose $\sigma_\tau \in \mathrm{Hom}_K(L, \overline{K})$ such that $\nu_{L'}(\sigma\tau(\pi_{L'}) - \pi_{L'}) \leq \nu_{L'}(\sigma(\alpha) - \sigma_\tau(\alpha)) - \deg(f)\lambda_n + 1$ for all $\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus \{\sigma_\tau\}$. Furthermore, we can take $\sigma_{\iota_{K'}} = \iota_L$. Let $S = \{\sigma_\tau\}_{\tau \in \mathrm{Hom}_K(K', \overline{K})}$. Note that $|S| \leq [K' : K] = \deg(\varphi_n)$.

We then obtain

$$\begin{aligned}
\mathrm{db}_K(\alpha) &\overset{(1)}{=} \sum_{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus \{\iota_L\}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) \\
&\geq \sum_{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus S} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) \\
&\overset{(2)}{=} \sum_{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus S} \left(\nu_L(\sigma(\alpha) - \alpha) - \sum_{\tau \in \mathrm{Hom}_K(K', \overline{K})} \nu_L(\sigma\tau(\pi_{L'}) - \pi_{L'}) \right) \\
&\overset{(3)}{\geq} \sum_{\sigma \in \mathrm{Hom}_K(L, \overline{K}) \setminus S} \left(\left(- \sum_{\tau \in \mathrm{Hom}_K(K', \overline{K}) \setminus \{\iota_{K'}\}} \nu_L(\sigma(\alpha) - \sigma_\tau(\alpha)) \right) + (\deg(f)\lambda_n - 1) \right) \\
&= \sum_{\tau \in \mathrm{Hom}_K(K', \overline{K}) \setminus \{\iota_{K'}\}} \left(-(\nu_L(f'(\sigma_\tau(\alpha))) + \sum_{\sigma \in S \setminus \{\sigma_\tau\}} \nu_L(\sigma(\alpha) - \sigma_\tau(\alpha))) \right) \\
&\qquad + (\deg(f) - |S|)(\deg(f)\lambda_n - 1) \\
&\overset{(4)}{=} (1 - \deg(\varphi_n))\deg(f)v_f(f') + \sum_{\substack{\tau \in \mathrm{Hom}_K(K', \overline{K}) \setminus \{\iota_{K'}\} \\ \sigma \in S \setminus \{\sigma_\tau\}}} \nu_L(\sigma(\alpha) - \sigma_\tau(\alpha)) \\
&\qquad + (\deg(f) - |S|)(\deg(f)\lambda_n - 1) \\
&\overset{(5)}{\geq} (1 - \deg(\varphi_n))\deg(f) \left(v_f(f') - (|S| - 1)\frac{\lambda_n}{\deg(\varphi_n)} \right) + (\deg(f) - |S|)(\deg(f)\lambda_n - 1) \\
&\overset{(6)}{\geq} (1 - \deg(\varphi_n))\deg(f) \left(v_f(f') - (\deg(\varphi_n) - 1)\frac{\lambda_n}{\deg(\varphi_n)} \right) + (\deg(f) - \deg(\varphi_n))(\deg(f)\lambda_n - 1) \\
&\overset{(7)}{\geq} (1 - \deg(\varphi_n))\deg(f) \left(e\lambda_n - \lambda_1 - (\deg(\varphi_n) - 1)\frac{\lambda_n}{\deg(\varphi_n)} \right) + (\deg(f) - \deg(\varphi_n))(\deg(f)\lambda_n - 1) \\
&= (\deg(\varphi_n)e(e-2) + e)\lambda_n + \deg(f)(\deg(\varphi_n) - 1)\lambda_1 - (e-1)\deg(\varphi_n) \\
&\overset{(8)}{\geq} (\deg(\varphi_n)e(e-2) + e)\lambda_n + e(\deg(\varphi_n) - 1) - (e-1)\deg(\varphi_n) \\
&= 2\deg(\varphi_n)\lambda_n + \deg(\varphi_n)(e^2 - 2e - 2)\lambda_n + \deg(\varphi_n) + e(\lambda_n - 1)
\end{aligned}$$

$$\overset{(9)}{>} 2\deg(\varphi_n)\lambda_n + (e^2 - 2e - 1)\deg(\varphi_n)$$
$$\overset{(10)}{\geq} 2\deg(\varphi_n)\lambda_n + (3e^2 - 6e - 3)$$
$$\overset{(11)}{\geq} 2\deg(\varphi_n)\lambda_n + 2e.$$

The non-labeled (in)equalities above follow from algebra and algebraic number theory. We justify the labeled ones. Equality (1) follows from Lemma 7.2. Equality (2) follows from Lemma 7.5. Inequality (3) follows from Lemma 7.22. Equality (4) follows from Corollary 4.12. Inequality (5) follows from Lemma 7.9(ii). Inequality (6) follows because $|S| \leq \deg(\varphi_n)$ and the overall coefficient of $|S|$ is $\deg(f)\lambda_n(\deg(\varphi_n)-1)/\deg(\varphi_n)-(\deg(f)\lambda_n-1)$, or $1-e\lambda_n$. Since $\lambda_n > 1$ (Lemma 4.3(v)), this coefficient is negative, so the expression is minimized when $|S| = \deg(\varphi_n)$. Inequality (7) follows from our assumption on $v_f(f')$. To prove inequality (8), note that $\deg(\varphi_2)\lambda_1 \geq 1$ by Lemma 4.3(iv) applied to v_2. So $\deg(f)\lambda_1 \geq e$, proving (8). Inequality (9) follows because $\lambda_n > 1$. Lastly, inequalities (10) and (11) follow because $\deg(\varphi_n) \geq 3$ and $e \geq 3$ respectively. □

7.4. **Inductive valuation length ≥ 2: the general case.** We maintain the assumption from §7.3 that the inductive length n of the valuation v_f is at least 2. It is *no longer assumed* that $\nu_K(\beta - \alpha)$ is constant as β ranges through the roots of φ_n, i.e., Assumption 7.8 is no longer in effect. However, we do fix a root β of φ_n and make the following assumption throughout §7.4.

Assumption 7.25. Among all its K-conjugates, β gives the *maximal* value of $\nu_K(\beta - \alpha)$.

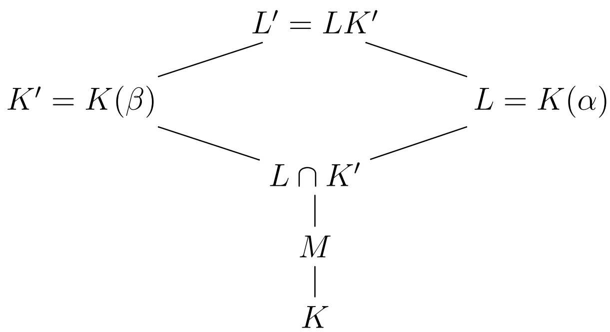
Notation for §7.4. We define the fields L, K', and L', as well as an important disk D below:

- $L = K(\alpha)$, $K' = K(\beta)$, and $L' = K'L = K(\alpha, \beta)$.
- D is the smallest disk containing α and β.
- $G := \{g \in \mathrm{Gal}(K^{\mathrm{sep}}/K) \mid g(D) = D\}$, and $M \subseteq K^{\mathrm{sep}}$ is the fixed field of G.
- $D = D_1, D_2, \ldots, D_r$ is the orbit of D under $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, say $\sigma_i(D) = D_i$ for certain representatives σ_i in $\mathrm{Gal}(K^{\mathrm{sep}}/K)$, $1 \leq i \leq r$.
- For $1 \leq i \leq r$, let $\beta_i = \sigma_i(\beta)$.

Lemma 7.26. *We have $M \subseteq L \cap K'$ and $\nu_M(\beta - \alpha)$ does not change when β is replaced by any of its M-conjugates.*

Proof. Since any element of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ that fixes α or β is in G, we conclude immediately that $M \subseteq L$ and $M \subseteq K'$. Furthermore, if β' is any M-conjugate of β, then $\beta' \in D$, which means that $\nu_M(\beta' - \alpha) \geq \nu_M(\beta - \alpha)$. Since $\nu_M(\beta - \alpha)$ is maximal by Assumption 7.25, $\nu_M(\beta' - \alpha) = \nu_M(\beta - \alpha)$. □

We thus have the diagram:



Lemma 7.27. *Let M be as in our notation for this section. Let f_M be the minimal polynomial for α over M, and let φ_M be the minimal polynomial for β over M. There exists $\lambda_M \in \mathbb{Q}_{\geq 0}$ such that f_M is a proper key polynomial over the valuation v of $M(x)$ corresponding via Proposition 4.6 to the diskoid $D(\varphi_M, \lambda_M)$ (i.e., $\{\gamma \in \overline{K} \mid \nu_M(\varphi_M(\gamma)) \geq \lambda_M\}$).*

Proof. By Proposition 4.9(iii), $D(\varphi_n, \lambda_n)$ is the smallest diskoid over K containing α and an element of lower K-degree. Note that it is also the smallest diskoid over K containing α and β, since $\nu_K(\varphi_n(\alpha)) = \lambda_n$ by Corollary 4.12. Since $M \subseteq L = K(\alpha)$, any element of lower M-degree than α also has lower K-degree than α. So $D(\varphi_n, \lambda_n)$ is also the smallest diskoid defined over K containing α and an element of lower M-degree. Since β is closest to α among its conjugates, this means the β approximates α as well as any other element of lower M-degree. Thus, the smallest diskoid D over M containing α and an element of lower M-degree contains β. By Proposition 4.8, $D = D(\varphi_M, \lambda_M)$ for some $\lambda_M \in \mathbb{Q}_{\geq 0}$. By Proposition 4.9(iii), f_M is a proper key polynomial over the valuation of $M(x)$ corresponding to this diskoid. ∎

Remark 7.28. Since $M \subseteq L \cap K'$, we have $e = \deg(f)/\deg(\varphi_n) = \deg(f_M)/\deg(\varphi_M)$.

Lemma 7.29. *Let M be as in our notation and let φ_M and λ_M be as in Lemma 7.27. Then*

$$\deg(\varphi_n)\lambda_n = \deg(\varphi_M)\lambda_M + \frac{\deg(\varphi_n)^2}{r} \sum_{i=2}^{r} \nu_K(\beta - \beta_i).$$

Proof. By Corollary 4.12, $\lambda_n = \nu_K(\varphi_n(\alpha)) = \sum \nu_K(\alpha - \beta')$, as β' ranges over all K-conjugates of β (i.e., the roots of φ_n). Similarly, $\lambda_M = \sum \nu_M(\alpha - \beta')$, as β' ranges over all M-conjugates of β (i.e., the roots of φ_M). Note that, since $M \subseteq K' = K(\beta)$, we have $[M : K] \deg(\varphi_M) = \deg(\varphi_n)$.

Let S_i be the set of all K-conjugates of β lying in the disk D_i. Then S_1 is the set of all M-conjugates of β. For the last line of the equality below, we need the following facts: First, $|S_i| = \deg(\varphi_n)/r$ for all i. Second, for $i \geq 2$, the definition of D shows that for any $\beta' \in S_i$, the distances between α and β and between β' and β_i are both smaller than the distance between α and β', so $\nu_K(\alpha-\beta')=\nu_K(\beta-\beta_i)$. So

$$\begin{aligned}
\deg(\varphi_n)\lambda_n &= \deg(\varphi_n)\sum_{\varphi_n(\beta')=0}\nu_K(\alpha-\beta')\\
&= \deg(\varphi_n)\sum_{\beta'\in S_1}\nu_K(\alpha-\beta')+\deg(\varphi_n)\sum_{i=2}^{r}\sum_{\beta'\in S_i}\nu_K(\alpha-\beta')\\
&= \deg(\varphi_M)\sum_{\beta'\in S_1}\nu_M(\alpha-\beta')+\deg(\varphi_n)\sum_{i=2}^{r}\sum_{\beta'\in S_i}\nu_K(\alpha-\beta')\\
&= \deg(\varphi_M)\lambda_M+\frac{\deg(\varphi_n)^2}{r}\sum_{i=2}^{r}\nu_K(\beta-\beta_i),
\end{aligned}$$

where the last equality follows because $\nu_K(\alpha-\beta')=\nu_K(\beta-\beta_i)$ for all $i\geq 2$. This completes the proof. □

Lemma 7.30. *Let M be as in Lemma 7.26 and φ_M and λ_M be as in Lemma 7.27. We have*

$$\mathrm{db}_M(\alpha)\geq\begin{cases}2\deg(\varphi_M)\lambda_M-1 & e=2\\ 2(\lfloor\deg(\varphi_M)\lambda_M\rfloor+e-1) & e>2.\end{cases}$$

If $M=K$ and $n\geq 2$, then equality holds only if $n=e=2$, φ_2 is Eisenstein, and $\deg(\varphi_2)$ is a power of $\operatorname{char} k$.

Proof. By Lemma 7.27, f_M is a proper key polynomial over the inductive valuation v corresponding to the diskoid $D(\varphi_M,\lambda_M)$. In particular, v can be written as $[v_{0,M},\, v_{1,M}(\varphi_{1,M})=\lambda_{1,M},\dots,v_{n_M}(\varphi_M)=\lambda_M]$, where $v_{0,M}$ is the Gauss valuation on $M(x)$ extending ν_M and n_M is the inductive length of v. By Lemma 7.26, all roots of φ_M are equidistant from α. By Remark 7.28, $\deg(f_M)/\deg(\varphi_M)=e$. Thus, with respect to M as our base field, we are in the situation of §7.2 when $n_M=1$ and in the situation of §7.3 when $n_M\geq 2$, with ν_M, f_M, φ_M, $\lambda_{1,M}$, λ_M, n_M, and e playing the roles of ν_K, f, φ_n, λ_1, λ_n, n, and e respectively.

If $n_M\geq 2$ and $\deg(f_M)$ is not a power of $\operatorname{char} k$, the lemma follows from Proposition 7.10. If $n_M\geq 2$ and $\deg(f_M)$ is a prime power, the lemma follows from Proposition 7.19 or 7.24, depending on which case of Proposition 7.18 we are in. Furthermore, equality holds only when $n_M=e=2$, $\varphi_{2,M}$ is Eisenstein, and $\deg(\varphi_{2,M})$ is a power of $\operatorname{char} k$. If $M=K$, this is the condition given in the lemma.

If $n_M=1$, we would like to apply Corollary 7.7, but we need to check its hypotheses. First, $\varphi_M=\varphi_{1,M}$, which is linear, so $\beta\in M$. By the definition of M, this means that $\nu_M(\alpha-\beta)>\nu_M(\beta-\beta')$ for any K-conjugate $\beta'\neq\beta$ of β. Since k is algebraically closed, there exists $\eta\in K$ such that $\nu_M(\beta-\eta)\geq 1$. The same is true for $\beta'-\eta$, so $\nu_M(\beta'-\beta)\geq 1$. We conclude that $\nu_M(\alpha-\beta)>1$. By Lemma 7.9(i), $\lambda_{1,M}>1$, which means that it can be written as $b_{1,M}/c_{1,M}$ in lowest terms with either $c_{1,M}=2$ or $c_{1,M}\geq 3$ and $b_{1,M}\geq 3$ (note that $c_{1,M}=e$, so $c_{1,M}\neq 1$). Now we can apply Corollary 7.7 to prove the lemma. □

We now prove the main result of §7. So that it stands alone as a reference, we repeat the notation from the beginning of §7 in the statement of the theorem.

Proposition 7.31. *Let $\alpha\in\mathcal{O}_{K^{\mathrm{sep}}}$ with positive valuation and minimal polynomial $f(x)$. Let*

$$v_f=[v_0,\, v_1(\varphi_1)=\lambda_1,\,\dots,\, v_n(\varphi_n)=\lambda_n]$$

be the unique Mac Lane valuation over which f is a proper key polynomial (Proposition 4.9(iv)). Let $e = \deg(f)/\deg(\varphi_n)$. Suppose $n \geq 2$. Then

$$\mathrm{db}_K(\alpha) \geq \begin{cases} 2\deg(\varphi_n)\lambda_n - 1 & e = 2 \\ 2(\lfloor \deg(\varphi_n)\lambda_n \rfloor + e - 1) & e > 2. \end{cases}$$

If equality holds, then $e = n = 2$, and φ_2 is Eisenstein.

Proof. Let M be as in our notation. By Lemma 7.2,

$$\begin{aligned} \mathrm{db}_K(\alpha) &= \sum_{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \setminus \{\iota_L\}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) \\ &= \sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \setminus \{\iota_L\} \\ \sigma|_M = \iota_M}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) + \sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \\ \sigma|_M \neq \iota_M}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) \\ &= \mathrm{db}_M(\alpha) + \sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \\ \sigma|_M \neq \iota_M}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)). \end{aligned}$$

By Lemma 7.30,

$$\mathrm{db}_M(\alpha) \geq \begin{cases} 2\deg(\varphi_M)\lambda_M - 1 & e = 2 \\ 2(\lfloor \deg(\varphi_M)\lambda_M \rfloor + e - 1) & e > 2 \end{cases}, \tag{7.32}$$

with equality as in Lemma 7.30. If $r = 1$, then $M = K$ and the proposition follows immediately. So assume $r \geq 2$. By Lemma 7.35 below,

$$\sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \\ \sigma|_M \neq \iota_M}} (\nu_L(\sigma(\alpha) - \alpha) - \nu_L(\sigma(\pi_L) - \pi_L)) \geq \frac{2\deg(\varphi_n)^2}{r} \sum_{i=2}^{r} \nu_K(\beta - \beta_i) + 2\epsilon, \tag{7.33}$$

where $\epsilon = 1$ if $e > 2$, and $\epsilon = 0$ if $e = 2$. Furthermore, if $e = 2$, then equality holds in (7.33) only if $n = 2$ and φ_2 is Eisenstein.

Adding (7.32) and (7.33), and using Lemma 7.29, we conclude that

$$\mathrm{db}_K(\alpha) \geq \begin{cases} 2\deg(\varphi_n)\lambda_n - 1 & e = 2 \\ 2(\deg(\varphi_n)\lambda_n + e - 1) + 2\lfloor \deg(\varphi_M)\lambda_M \rfloor - 2\deg(\varphi_M)\lambda_M + 2\epsilon & e > 2 \end{cases} \tag{7.34}$$

with equality holding only if it does in (7.33). If $e = 2$, then we are done. If $e > 2$, then $\epsilon = 1$ and (7.34) proves the theorem, indeed, with a strict inequality. ☐

Lemma 7.35. *Inequality (7.33) is true. Furthermore, if equality holds, then $n = 2$ and φ_2 is Eisenstein.*

Proof. If $r = 1$, then (7.33) is just $0 = 0$, so assume $r > 1$. For $1 \leq i \leq r$, let $H_i \subseteq \mathrm{Hom}_K(L, \overline{K})$ be the subset of embeddings extending to an automorphism of $\overline{K}$ that sends D to D_i. Each H_i has cardinality $\deg(f)/r$. The sum on the left-hand side of (7.33) is taken over the union of the H_i, $2 \leq i \leq r$. Furthermore, since all elements of D_i ($2 \leq i \leq r$) are equidistant from all elements of D, we have that for $\sigma \in H_i$,

$$\nu_L(\sigma(\alpha) - \alpha) = \nu_L(\beta - \beta_i)$$

when $2 \le i \le r$. Thus

$$\sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\alpha) - \alpha) = \frac{\deg(f)}{r} \sum_{i=2}^{r} \nu_L(\beta - \beta_i). \tag{7.36}$$

On the other hand, using Lemma 7.5 twice, we have

$$\begin{aligned} \sum_{\substack{\sigma \in \mathrm{Hom}_K(L,\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\pi_L) - \pi_L) &= \sum_{\substack{\sigma \in \mathrm{Hom}_K(L',\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\pi_{L'}) - \pi_{L'}) \\ &= \sum_{\substack{\sigma \in \mathrm{Hom}_K(K',\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\pi_{K'}) - \pi_{K'}) \end{aligned} \tag{7.37}$$

Now, there are exactly $\deg(\varphi_n)/r$ elements of $\mathrm{Hom}_K(K', \overline{K})$ that extend to an automorphism of $\overline{K}$ taking D to D_i for each i, and again, the sum in (7.37) is over those taking D to D_i, for $2 \le i \le r$. For the same reason as before, for any σ taking D to D_i, we have $\nu_L(\sigma(\beta) - \beta) = \nu_L(\beta - \beta_i)$. Thus, using Lemma 7.3(ii), and grouping together K-embeddings by where their extensions to $\overline{K}$ send D,

$$\begin{aligned} \sum_{\substack{\sigma \in \mathrm{Hom}_K(K',\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\pi_{K'}) - \pi_{K'}) &\le \sum_{\substack{\sigma \in \mathrm{Hom}_K(K',\overline{K}) \\ \sigma|_M \neq \iota_M}} \nu_L(\sigma(\beta) - \beta) \\ &= \frac{\deg(\varphi_n)}{r} \sum_{i=2}^{r} \nu_L(\beta - \beta_i), \end{aligned} \tag{7.38}$$

with equality only when there exists $\eta \in K$ such that $\beta - \eta$ is a uniformizer of K'. Since $\nu_K(\beta) > 0$, we have that β is in fact a uniformizer of K', so φ_n is Eisenstein. By Remark 5.31 and the uniqueness of inductive valuation length, $n = 2$.

From (7.36) and (7.38), we have that the left-hand side of (7.33) is at least

$$\frac{\deg(f) - \deg(\varphi_n)}{r} \sum_{i=2}^{r} \nu_L(\beta - \beta_i),$$

which in turn equals

$$\frac{e(e-1)\deg(\varphi_n)^2}{r} \sum_{i=2}^{r} \nu_K(\beta - \beta_i). \tag{7.39}$$

If $e = 2$, then (7.33) clearly holds. If $e > 2$, we need only check that

$$\frac{(e(e-1) - 2)\deg(\varphi_n)^2}{r} \sum_{i=2}^{r} \nu_K(\beta - \beta_i) > 2$$

whenever $r > 1$. But $\nu_K(\beta - \beta_i) \ge 1/\deg(\varphi_n)$ and $\deg(\varphi_n) \ge r$, so this is clear. □

8. Comparing number of components and the discriminant bonus

In this short section, we use the results of the previous two sections to prove a key inequality that will be used in the proof of the conductor-discriminant inequality for irreducible f in §9 and for reducible f in §10. Let $\alpha \in \mathcal{O}_{K^{\mathrm{sep}}}$ with positive valuation, and let f be the minimal polynomial of α. Recall (Definition 6.1) that if $\deg(f) \geq 2$, we have constructed a regular model $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ of $\mathbb{P}^1_K$ on which the horizontal divisor D_α is regular (Theorem 6.9). Recall the construction: $v_f := [v_0,\, v_1(\varphi_1) = \lambda_1, \dots,\, v_n(\varphi_n) = \lambda_n]$ is the unique Mac Lane valuation over which f is a proper key polynomial. If $n \geq 1$, there is a valuation $v_f' = [v_0,\, v_1(\varphi_1) = \lambda_1, \dots,\, v_n(\varphi_n) = \lambda']$ that is the *successor* valuation to v_f, and $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ is the minimal regular resolution of the v_f'-model of $\mathbb{P}^1_K$.

Let $\mathcal{Y}_{f,0}'$ be the minimal regular resolution of $\mathcal{Y}_{v_f'}^{\mathrm{reg}}$ including v_0 (Corollary 5.16). We also constructed a regular model $\mathcal{Z}_{f,0}$ of $\mathbb{P}^1_K$ on which D_α is regular, which dominates $\mathcal{Y}_{f,0}'$ (Definition 6.1). Let $N_{\mathcal{Y}_{f,0}'}$ (resp. $N_{\mathcal{Z}_{f,0}}$) be the number of irreducible components of the special fiber of $\mathcal{Y}_{f,0}'$ (resp. $\mathcal{Z}_{f,0}$). In Proposition 6.20, we placed upper bounds on $N_{\mathcal{Y}_{f,0}'}$ and $N_{\mathcal{Z}_{f,0}}$. Write $L = K(\alpha)$ and $K' = K(\beta)$, and let $e = \deg(f)/\deg(\varphi_n)$. Recall that $\mathrm{db}_K(\alpha)$ is the discriminant bonus of α over K.

Theorem 8.1. *There is a regular model of $\mathbb{P}^1_K$ on which D_α is regular, which includes v_0, and whose special fiber has at most $1 + \mathrm{db}_K(\alpha)/2$ irreducible components. In fact, we can take this model as follows:*

- (i) *If $n = 0$ (this means f is a linear polynomial) or if f is Eisenstein, then we can take the v_0-model of $\mathbb{P}^1_K$.*
- (ii) *If $n = 1$, $\deg(f) \geq 3$ is odd, and $\lambda_1 = 2/\deg(f)$ (this means all roots of f have valuation $2/\deg(f)$), we can take the model from Corollary 6.11.*
- (iii) *If $e = 2$, then we can take $\mathcal{Y}_{f,0}'$.*
- (iv) *In all other cases, we can take $\mathcal{Z}_{f,0}$.*

Furthermore, suppose we are in case (iv) above. The number of irreducible components on the special fiber of $\mathcal{Z}_{f,0}$ is strictly less than $1 + \mathrm{db}_K(\alpha)/2$ unless $n = 1$ and $\lambda_n = 3/4$ or $4/3$.

Proof. If $n = 0$, then f is a key polynomial over v_0, which means f is linear and $\alpha \in K$ (Lemma 4.3(i)). By Lemma 6.14, D_α is regular on the v_0-model of $\mathbb{P}^1_K$, which has one irreducible component. Also, $\mathrm{db}_K(\alpha) = 0$. This proves part (i) for $n = 0$.

Suppose $n = 1$, and write $v_f = [v_0,\, v_1(\varphi_1) = \lambda_1 = b/c]$, with b/c in lowest terms. Note that $c = \deg(f) = \deg(f)/\deg(\varphi_1) = e$ (Lemma 4.3(iii)). In particular, $c \geq 2$ and $b \geq 1$. By Corollary 7.4, $\mathrm{db}_K(\alpha) \geq (c-1)(b-1)$.

If $b = 1$, then f is Eisenstein. Using Remark 5.31 one sees that $v_f' = v_0$, so $\mathcal{Y}_{v_f'}^{\mathrm{reg}} = \mathcal{Y}_{f,0}'$ is simply the v_0-model of $\mathbb{P}^1_K$. So $N_{\mathcal{Y}_{f,0}'} = 1$, proving the Eisenstein case of (i). If $b = 2$, then Corollary 6.11 shows that D_α is regular on a model of $\mathbb{P}^1_K$ containing $(c+1)/2 = 1 + (c-1)/2$ irreducible components, and this model includes v_0. Since $\mathrm{db}_K(\alpha) \geq (c-1)(b-1) = c-1$, part (ii) is proved.

So assume $b \geq 3$. By Proposition 6.20, $N_{\mathcal{Y}_{f,0}'} \leq \deg(\varphi_1)\lambda_1 + \frac{1}{2}$ when $e = 2$, and $N_{\mathcal{Z}_{f,0}} \leq \lfloor \deg(\varphi_1)\lambda_1 \rfloor + e$ when $e > 2$. When $e = 2$, this falls under part (iii) of the theorem, and the result follows from Corollary 7.7. When $e > 2$, this falls under part (iv) of the theorem, and again the result follows from Corollary 7.7. Since equality holds in Corollary 7.7 only when $b = 3$ or $\lambda_1 = 4/3$, we can have $N_{\mathcal{Z}_{f,0}} = 1 + \mathrm{db}_K(\alpha)/2$ only in this case. But if $b = 3$ and $c \geq 5$, the strengthening of Proposition 6.20 mentioned in its statement precludes equality. So the last clause of the theorem is proved whenever $n = 1$.

Now assume $n \geq 2$. Again, by Proposition 6.20, $N_{\mathcal{Z}_{f,0}} \leq \lfloor \deg(\varphi_n)\lambda_n \rfloor + e$ when $e > 2$, and $N_{\mathcal{Y}'_{f,0}} \leq \deg(\varphi_n)\lambda_n + \frac{1}{2}$ when $e = 2$. In both cases, the theorem (including the fact that $N_{\mathcal{Z}_{f,0}} = 1 + \mathrm{db}_K(\alpha)/2$ is impossible except when $e = 2$), follows immediately from Proposition 7.31. □

Remark 8.2. Note that, in parts (ii) and (iii) of Theorem 8.1, the model that we use arises from contracting *exactly one* component of $\mathcal{Z}_{f,0}$. In part (ii), this follows from Remark 6.13. In part (iii), this follows from Lemma 6.21.

Remark 8.3. The model from Theorem 8.1 may not be the model that realizes the conductor-discriminant inequality even when f is irreducible, since we need a model where all odd components of $\mathrm{div}(f)$ (both horizontal and *vertical*) are not only regular but also *pairwise disjoint*. In the following sections, we will make the necessary modifications to the models coming from this section to separate all odd components of $\mathrm{div}(f)$ as well as handle multiple irreducible factors.

9. Proof of the conductor-discriminant inequality in the irreducible case

Let $f \in \mathcal{O}_K[x]$ be a polynomial satisfying Assumption 2.1, all of whose roots have positive valuation. For any regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$, write $\mathcal{Y}_s$ for its special fiber, and write $N_{\mathcal{Y}}$ (resp. $N_{\mathcal{Y},\mathrm{odd}}$, $N_{\mathcal{Y},\mathrm{even}}$) for the number of irreducible components on $\mathcal{Y}_s$ (resp. the number of such components on which f has odd, resp. even order). Recall from (2.10) that in order to prove the conductor-discriminant inequality for f, it suffices to find $\mathcal{Y}$ as above satisfying

$$\mathrm{db}_K(f) \geq 2(N_{\mathcal{Y}} - 1 - N_{\mathcal{Y},\mathrm{odd}}) = 2(N_{\mathcal{Y},\mathrm{even}} - 1), \tag{9.1}$$

such that the odd components of $\mathrm{div}_0(f)$ are regular on $\mathcal{Y}$ and do not intersect each other.

After stating a couple of lemmas in §9.1, we prove inequality (9.1) in §9.2 when f is *irreducible* over $K[x]$. This proof is relatively straightforward, given Theorem 8.1.

Recall that if α is a root of f and $\mathcal{Y}$ is a normal model of $\mathbb{P}^1_K$ that is clear from context, then D_α is the subscheme of $\mathcal{Y}$ given by the closure of $\alpha \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$.

9.1. Preliminary lemmas.

Lemma 9.2. *Let $g \in K(x)^\times$, let $\mathcal{W}$ be a regular model of $\mathbb{P}^1_K$, let $\mathcal{V}$ be the minimal modification of $\mathcal{W}$ such that no odd horizontal component of $\mathrm{div}_0(g)$ meets any odd vertical component of $\mathrm{div}(g)$, and let $\mathcal{Y}$ be the minimal modification of $\mathcal{V}$ such that such that no two odd vertical components of $\mathrm{div}(g)$ meet. Let $N_{\mathcal{Y},\mathrm{even}}$ (resp. $N_{\mathcal{V}}$, $N_{\mathcal{W}}$) be the number of even irreducible vertical components for $\mathrm{div}(g)$ on $\mathcal{Y}$ (resp. the total number of irreducible components on the special fiber of $\mathcal{V}$, $\mathcal{W}$). If $\mathrm{div}(g)$ has an odd component on $\mathcal{V}$, then $N_{\mathcal{Y},\mathrm{even}} \leq N_{\mathcal{V}} - 1$. If not, then $\mathcal{V} = \mathcal{W}$ and $N_{\mathcal{Y},\mathrm{even}} = N_{\mathcal{V}} = N_{\mathcal{W}}$.*

Proof. Let $N_{\mathcal{V},\mathrm{odd}}$ (resp. $N_{\mathcal{V},\mathrm{even}}$) be is the number of odd (resp. even) vertical components for $\mathrm{div}(g)$ on $\mathcal{V}$. Assume first that $N_{\mathcal{V},\mathrm{odd}} \geq 1$. Since the special fiber of $\mathcal{V}$ is a tree, the number of intersection points of two odd vertical components of $\mathrm{div}(g)$ on $\mathcal{V}$ is at most $N_{\mathcal{V},\mathrm{odd}} - 1$. By assumption, no odd horizontal component of $\mathrm{div}(g)$ meets any of these intersection points on $\mathcal{V}$. By Lemma 2.11 and Remark 2.12, blowing up each of these intersection points yields an exceptional divisor that is an even vertical component for $\operatorname{div}(g)$. The strict transforms of the even (resp. odd) vertical components of $\operatorname{div}(g)$ on $\mathcal{V}$ remain even (resp. odd) on $\mathcal{Y}$. So $N_{\mathcal{Y},\text{even}} \leq (N_{\mathcal{V},\text{odd}} - 1) + N_{\mathcal{V},\text{even}} = N_{\mathcal{V}} - 1$.

If $N_{\mathcal{V},\text{odd}} = 0$, then $N_{\mathcal{W},\text{odd}} = 0$ as well. By the definition of $\mathcal{V}$ and $\mathcal{Y}$, we have $\mathcal{W} = \mathcal{V} = \mathcal{Y}$, and $N_{\mathcal{Y},\text{even}} = N_{\mathcal{V}} = N_{\mathcal{W}}$. □

Lemma 9.3. *Suppose $g \in K[x]$ is a separable polynomial with a root γ of positive valuation, and let h be the minimal polynomial of γ. Let $\mathcal{W}$ be a regular model of $\mathbb{P}^1_K$, and assume that D_γ does not meet any other horizontal components of $\operatorname{div}(g)$ on $\mathcal{W}$. If $\deg(h) \geq 2$ and $\mathcal{W}$ includes v_h, or if h is linear, then at most one closed point blowup is required to ensure that D_γ does not meet any odd vertical components of $\operatorname{div}(g)$.*

Proof. If $\deg(h) \geq 2$ and $\mathcal{W}$ includes v_h, then by Lemma 5.27(i), D_γ meets a unique component $\overline{W}$ on the special fiber of $\mathcal{W}$, say at the point P. If g is linear, the same is true by [LL99, Lemma 5.1(a)].

If $\overline{W}$ is an even component for $\operatorname{div}(g)$, then there is nothing to show. If not, then note that by our assumption, $\operatorname{div}(g)$ in $\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{W},P}$ consists solely of an odd vertical component and an odd horizontal component (the restriction of D_γ). By Lemma 2.11 and Remark 2.12, blowing up P yields an exceptional divisor E which is even for $\operatorname{div}(g)$. Once again by Lemma 5.27(i), the component E is the unique component that D_γ meets. □

9.2. **The Proof.** Throughout §9.2, we let $f = \pi_k^b f_1$, where f_1 is monic and irreducible with a root α of positive valuation, and $b \in \{0, 1\}$. Assume $\deg(f) \geq 2$. Let v_f be the unique Mac Lane valuation over which f is a proper key polynomial, and let $\mathcal{W}$ be the regular model of $\mathbb{P}^1_K$ from Theorem 8.1.

Theorem 6.9 shows that D_α is regular on $\mathcal{W}$. Since f is irreducible over $K(x)$, the divisor D_α is the only odd horizontal component of $\operatorname{div}_0(f)$. Let $\mathcal{V}$ be the the minimal blowup of $\mathcal{W}$ such that this horizontal component intersects only even components of $\operatorname{div}(f)$, and let $\mathcal{Y}$ be the minimal blowup of $\mathcal{V}$ such that the odd vertical components of $\operatorname{div}(f)$ are disjoint. Let $N_{\mathcal{W}}$, $N_{\mathcal{V}}$, and $N_{\mathcal{Y}}$ be the number of irreducible components of the special fiber of $\mathcal{W}$, $\mathcal{V}$, and $\mathcal{Y}$, respectively, and let $N_{\mathcal{Y},\text{even}}$ be the number of even vertical components of $\operatorname{div}(f)$ on $\mathcal{Y}$. By construction, the odd components of $\operatorname{div}_0(f)$ are regular and disjoint on $\mathcal{Y}$.

The conductor-discriminant inequality for f is a consequence of the following theorem.

Theorem 9.4. *Let $f = \pi_K^b f_1$, and let α, $\mathcal{W}$, $\mathcal{V}$, and $\mathcal{Y}$ be as above. Then $\mathcal{Y}$ satisfies (9.1). That is, the conductor-discriminant inequality holds for all f satisfying Assumption 2.1 with $r = 1$ and all roots of positive valuation.*

Proof. By Theorem 8.1, $\operatorname{db}_K(\alpha) \geq 2(N_{\mathcal{W}} - 1)$. Now, if $\mathcal{W}$ has no odd vertical components for $\operatorname{div}(f)$, then Lemma 9.2 shows that $\mathcal{W} = \mathcal{V} = \mathcal{Y}$, and thus $\operatorname{db}_K(f) \geq \operatorname{db}_K(\alpha) \geq 2(N_{\mathcal{W}} - 1) = 2(N_{\mathcal{Y},\text{even}} - 1)$, and we are done. So assume $\mathcal{W}$ has at least one odd vertical component for $\operatorname{div}(f)$. We claim that $\operatorname{db}_K(f) \geq 2(N_{\mathcal{V}} - 2)$. If we admit this claim, Lemma 9.2 shows that $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y},\text{even}} - 1)$, which is (9.1), finishing the proof.

To prove the claim, we go case by case through the possibilities for f_1 in Theorem 8.1. The linear part of case (i) does not occur since we are assuming that $\deg(f) \geq 2$. So assume f_1 is Eisenstein of degree d. If $b = 0$, then v_0 is an even component for $\operatorname{div}_0(f)$, so $\mathcal{W} = \mathcal{V}$. Since $N_{\mathcal{W}} = 1$, the inequality $\operatorname{db}(f) \geq 2(N_{\mathcal{V}} - 1)$ follows immediately. Now suppose $b = 1$. By

Remark 5.31, we have $v_{f_1} = [v_0,\, v_1(x) = 1/d]$. If $\mathcal{W}'$ is the minimal regular resolution of the v_{f_1}-model of $\mathbb{P}^1_K$, then Lemma 5.27 implies that D_α meets only the component correpsonding to v_{f_1} on the special fiber of $\mathcal{W}'$. Now, $v_{f_1}(f) = v_{f_1}(\pi_K) + v_{f_1}(f) = 2$, so this component is even for $\mathrm{div}(f)$. Thus $\mathcal{W}'$ dominates $\mathcal{V}$ (in fact, they are equal, but we don't need this). Now, $\mathcal{W}'$ includes exactly the valuations $v_\lambda := [v_0,\, v_1(x) = \lambda]$ for $\lambda \in \{0, 1, 1/2, 1/3, \ldots, 1/d\}$, as can be computed, e.g., from Proposition 5.12. So $N_{\mathcal{V}} \leq N_{\mathcal{W}'} = d + 1$. On the other hand, $\mathrm{db}_K(f) = 2(d-1) + \mathrm{db}_K(f_1) = 2(d-1)$. So we have $\mathrm{db}_K(f) \geq 2(N_{\mathcal{V}} - 2)$. This finishes case (i).

In case (ii), $\mathcal{W}$ is the model from Corollary 6.11 and $\mathrm{db}_K(\alpha) = 2(N_{\mathcal{W}} - 1)$. Let $w := [v_0,\, v_1(x) = 1/((c-1)/2)]$. By Corollary 5.4, D_α meets the intersection of the two components of the special fiber of the $\{v_0, w\}$-model of $\mathbb{P}^1_K$, and therefore also in the $\mathcal{W}$ model of $\mathbb{P}^1_K$, since the natural map from $\mathcal{W}$ to the $\{v_0, w\}$-model is an isomorphism above the point where v_0 and w meet. If $b = 0$, one checks that $v_0(f) = 0$ and $w(f) = 2$, so these are both even components for f. Thus $\mathcal{V} = \mathcal{W}$ and $\mathrm{db}_K(f) = \mathrm{db}(f_1) = 2(N_{\mathcal{V}} - 1)$, proving the claim. If $b = 1$, then $\mathrm{db}_K(f) = 2(c-1) + \mathrm{db}_K(\alpha) = 2(N_{\mathcal{W}} + c - 2)$. Now, the minimal regular resolution $\mathcal{W}'$ of the $v_{f_1} = [v_0,\, v_1(x) = 2/c]$-model of $\mathbb{P}^1_K$ is constructed in Remark 6.13, and has two more irreducible components on its special fiber than $\mathcal{W}$. By Lemma 9.3, $N_{\mathcal{V}} \leq N_{\mathcal{W}'} + 1 = N_{\mathcal{W}} + 3$, which implies that $\mathrm{db}_K(f) \geq 2(N_{\mathcal{V}} + c - 5) \geq 2(N_{\mathcal{V}} - 2)$, since $c \geq 3$. This finishes case (ii).

In case (iii), Corollary 6.17(ii) and Proposition 6.3(iv) show that D_α meets only the v_f'-component of the special fiber of $\mathcal{W}$, where v_f' is the successor valuation of v_f. This is an even component of $\mathrm{div}(f)$ by Lemma 6.22, regardless of whether $b = 0$ or $b = 1$. So $\mathcal{V} = \mathcal{W}$, and we have $\mathrm{db}_K(f) \geq \mathrm{db}_K(\alpha) \geq 2(N_{\mathcal{V}} - 1)$. This finishes case (iii).

In case (iv), we have $\mathcal{W} = \mathcal{Z}_{f_1,0}$. By the definition of $\mathcal{Z}_{f_1,0}$, it requires only one closed point blowup to obtain a model including v_{f_1}. By Lemma 9.3, $N_{\mathcal{V}} \leq N_{\mathcal{W}} + 2$. This means that $\mathrm{db}_K(\alpha) \geq 2(N_{\mathcal{W}} - 1) \geq 2(N_{\mathcal{V}} - 3)$. If $b = 1$, then since $\deg(f) \geq 2$, we have $\mathrm{db}_K(f) \geq \mathrm{db}_K(\alpha) + 2 \geq 2(N_{\mathcal{V}} - 2)$. If $b = 0$, then unless $n = 1$ and either $\lambda_1 = 3/4$ or $4/3$, Theorem 8.1 actually implies that $\mathrm{db}_K(\alpha) \geq 2N_{\mathcal{W}}$. So $\mathrm{db}_K(\alpha) \geq 2(N_{\mathcal{V}} - 2)$, which is the estimate we need.

The last remaining case is when $b = 0$ and $v_{f_1} = [v_0,\, v_1(\varphi_1(x)) = \lambda]$ with $\lambda \in \{3/4, 4/3\}$. Again, it requires one closed point blowup to obtain a model $\mathcal{W}'$ including v_{f_1}. By Lemma 5.27, D_α meets only this component. If we show that this is an even component of $\mathrm{div}(f)$, then $\mathcal{W}'$ dominates $\mathcal{V}$, and thus $N_{\mathcal{V}} \leq N_{\mathcal{W}} + 1$. So $\mathrm{db}_K(f) = \mathrm{db}_K(\alpha) \geq 2(N_{\mathcal{W}} - 1) \geq 2(N_{\mathcal{V}} - 2)$, which is the inequality we want. If $\lambda = 3/4$, then Lemma 4.3(ii) shows that $v_{f_1}(f_1) = 3$, so $e(v_{f_1}/v_0)v_{f_1}(f_1) = 4 \cdot 3$ which is even. If $\lambda = 4/3$, then by Lemma 4.3, $v_{f_1}(f_1) = 4$, so $e(v_{f_1}/v_0)v_{f_1}(f_1) = 3 \cdot 4$ which is even. We have completed case (iv). □

10. Proof of the conductor-discriminant inequality in the reducible case

10.1. **Setup/notation.** We now come to the proof of the conductor-discriminant inequality for a reducible polynomial. Assume throughout §10 that

$$f = \pi_K^b f_1 \cdots f_r \tag{10.1}$$

with $b \in \{0, 1\}$, and the f_i are pairwise distinct monic irreducible separable polynomials over K, all of whose roots have positive valuation. That is, f satisfies Assumption 2.1 and additionally has all roots of positive valuation. Assume further that $r \geq 2$. Recall from (2.6)

that
$$\mathrm{db}_K(f) = \sum_{i=1}^{r} \mathrm{db}_K(f_i) + \sum_{1\le i<j\le r} 2\rho_{f_i,f_j} + 2b(\deg(f)-1).$$

Suppose $1 \le i \le r$. If f_i has degree at least 2, recall that $v_{f_i} = [v_0, v_{1,i}(\varphi_{1,i}(x)) = \lambda_{1,i}, \ldots, v_{n_i,i}(\varphi_{n_i,i}(x)) = \lambda_{n_i,i}]$ is the unique Mac Lane valuation over which f_i is a proper key polynomial, and v'_{f_i} is its successor valuation (§5.3). We define $\mathcal{Y}'_{f_i,0}$ and $\mathcal{Z}_{f_i,0}$ as in Definition 6.1. By Proposition 5.25(ii), $\mathcal{Y}'_{f_i,0}$ is a (possibly trivial) contraction of $\mathcal{Z}_{f_i,0}$.

Definition 10.2. For $1 \le i \le r$, we define a model $\mathcal{W}_{f_i}$ of $\mathbb{P}^1_K$ as follows:

- If f_i is linear or Eisenstein, we define $\mathcal{W}_{f_i}$ to be the v_0-model of $\mathbb{P}^1_K$.
- Otherwise, we define $\mathcal{W}_{f_i}$ to be $\mathcal{Z}_{f_i,0}$.

Remark 10.3.

(i) Since Remark 5.31 shows that $v'_{f_i} = v_0$ when f_i is Eisenstein, and Remark 6.2 shows that $\mathcal{Z}_{f_i,0}$ dominates $\mathcal{Y}^{\mathrm{reg}}_{v'_{f_i}}$, it follows that $\mathcal{W}_{f_i}$ is a regular model including v'_{f_i} and v_0 in all cases when $\deg(f_i) \ge 2$. By Corollary 6.10 and Lemma 6.14, the divisor $\mathrm{div}_0(f_i)$ is regular on $\mathcal{W}_{f_i}$ for all i, regardless of the degree of f_i. By Lemma 6.16, $\mathcal{W}_{f_i}$ is the v_0-model if and only if f_i is linear or Eisenstein.

(ii) It is not possible for $\mathcal{W}_{f_i}$ to include v_0 and exactly one other valuation. This is because, if $\mathcal{W}_{f_i}$ includes more than one valuation, then by definition, f_i is neither Eisenstein nor linear, and by Remark 6.2, $\mathcal{W}_{f_i} = \mathcal{Z}_{f_i,0}$ includes v'_{f_i} and v''_{f_i}. Since $v_0 \preceq v'_{f_i} \prec v''_{f_i}$ by definition, the only way for $\mathcal{W}_{f_i}$ to include v_0 and exactly one other valuation is if $v'_{f_i} = v_0$. However, if $v'_{f_i} = v_0$, then Lemma 6.16 shows that f_i is linear or Eisenstein, a contradiction.

(iii) The model $\mathcal{W}_{f_i}$ is exactly the model used in Theorem 8.1 except when either
- f_i has all roots of valuation $2/c$ with $c \ge 3$ odd,
- or $\deg(f_i) = 2\deg(\varphi_{n_i,i})$ and f_i is non-Eisenstein (this is the $e = 2$ case of Theorem 8.1).

In both of these cases, Remark 8.2 shows that $\mathcal{W}_{f_i} = Z_{f_i,0}$ includes one more valuation than the model used in Theorem 8.1. Combining this with Theorem 8.1, $N_{\mathcal{W}_{f_i}} \le 2+\mathrm{db}_K(f_i)/2$, where $N_{\mathcal{W}_{f_i}}$ is the total number of irreducible components of the special fiber of $\mathcal{W}_{f_i}$. Note that in both of these cases, $\mathcal{W}_{f_i}$ is neither linear nor Eisenstein.

Definition 10.4. For $f = \pi_K^b f_1 \cdots f_r$ as in (10.1), we define the following regular models of $\mathbb{P}^1_K$ arising from f (cf. §9.2):

- The model $\mathcal{W}_f$ is the minimal regular model of $\mathbb{P}^1_K$ dominating all the $\mathcal{W}_{f_i}$ such that the horizontal parts of $\mathrm{div}_0(f_i)$ and $\mathrm{div}_0(f_j)$ do not meet when $i \neq j$.
- The model $\mathcal{V}_f$ is the minimal regular model of $\mathbb{P}^1_K$ dominating $\mathcal{W}_f$ such that no horizontal component of $\mathrm{div}_0(f)$ intersects any odd vertical component of $\mathrm{div}(f)$.
- The model $\mathcal{Y}_f$ is the minimal regular model of $\mathbb{P}^1_K$ dominating $\mathcal{V}_f$ on which no two odd vertical components of $\mathrm{div}(f)$ meet.

Remark 10.5. Since each of the $\mathcal{W}_{f_i}$ dominates the v_0-model, so do $\mathcal{W}_f, \mathcal{V}_f, \mathcal{Y}_f$.

Note that on any of these models, the odd vertical components of $\mathrm{div}(f)$ are the same as those of $\mathrm{div}_0(f)$. We write $N_{\mathcal{W}_{f_i}}$ (resp. $N_{\mathcal{W}_f}$, $N_{\mathcal{V}_f}$, $N_{\mathcal{Y}_f}$) for the number of irreducible components on the special fiber of $\mathcal{W}_{f_i}$ (resp. $\mathcal{W}_f$, $\mathcal{V}_f$, $\mathcal{Y}_f$). By construction, the odd components of $\operatorname{div}(f)$ (ignoring the possible horizontal component at ∞) are regular and disjoint on $\mathcal{Y}_f$. If $N_{\mathcal{Y}_f,\text{even}}$ is the number of even vertical components of $\operatorname{div}(f)$ on $\mathcal{Y}_f$, then the conductor-discriminant inequality for f will be proven if $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y}_f,\text{even}} - 1)$ as in (9.1).

10.2. **Brief overview of the rest of the proof.** We already know from Theorem 8.1 that $\operatorname{db}(f_i) \geq 2(N_{\mathcal{W}_{f_i}} - 1)$ unless $\mathcal{W}_{f_i}$ is in one of the exceptional cases in Remark 10.3(iii). Now, by Lemma 5.5, the minimal regular model dominating all the $\mathcal{W}_{f_i}$ includes the union of the valuations included in each $\mathcal{W}_{f_i}$. Since one of those valuations is v_0 by Remark 10.5, this model includes at most $\sum_i (N_{\mathcal{W}_{f_i}} - 1)$ valuations, in addition to v_0. In §10.3, we show that separating D_{α_i} from D_{α_j} on this model requires not more than ρ_{f_i,f_j} closed point blowups (see Proposition 10.8, which proves somewhat better results in certain cases). Since $\mathcal{W}_f$ is constructed by separating all D_{α_i} from each other, we show in §10.4 that

$$N_{\mathcal{W}_f} - 1 \leq \sum_i (N_{\mathcal{W}_{f_i}} - 1) + \sum_{i<j} \rho_{f_i,f_j}$$

(Corollary 10.11 actually yields a slightly better bound, since we keep track of some redundancies in the necessary blowups). Notice that the right-hand side above is bounded above by $\operatorname{db}(f)$ (as long as we ignore the exceptional cases in Remark 10.3(iii)).

In §10.5, we bound the number of additional blowups needed to construct $\mathcal{V}_f$ from $\mathcal{W}_f$. Constructing $\mathcal{V}_f$ is the process of separating the D_{α_i} from the odd vertical components of $\operatorname{div}(f)$. Usually, this will only require 2 blowups for each f_i (one to ensure that the model includes v_{f_i}, and then one more using Lemma 9.3), but there are some exceptions. These blowups can be accounted for in the "savings" from Corollary 10.11, and we thus get an upper bound for $N_{\mathcal{V}_f}$ similar to that for $N_{\mathcal{W}_f}$.

Lastly, in §10.6, we combine our estimate for $N_{\mathcal{V}_f}$ with Lemma 9.2 to attain the desired bound on $N_{\mathcal{Y}_{\text{even}}}$.

10.3. **Obtaining $\mathcal{W}_f$ from the $\mathcal{W}_{f_i}$ — two irreducible polynomial factors.** We start with the case $r = 2$ in (10.1).

Lemma 10.6. *Let $g(x), h(x) \in K[x]$ be irreducible and monic, having roots α_g, α_h of positive valuation respectively. Assume $\deg(g) \geq 2$ so that the valuations v_g, v_g', and v_g'' from § 5.4 and Notation 5.30 are well-defined. Recall that v_g is the unique Mac Lane valuation over which g is a proper key polynomial, and similarly write v_h for the unique Mac Lane valuation over which h is a proper key polynomial. Write*

$$\begin{aligned} v_g &:= [v_0,\, v_1(\varphi_1(x)) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n] \\ v_g' &:= [v_0, \ldots,\, v_n(\varphi_n) = \lambda'] \\ v_g'' &:= [v_0, \ldots,\, v_n(\varphi_n) = \lambda'']. \end{aligned}$$

Then D_{α_g} and D_{α_h} meet on $\mathcal{Z}_{g,0}$ only if v_h is a Mac Lane valuation of the form

$$[v_0, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n(\varphi_n) = \lambda, \ldots,\, v_m(\varphi_m) = \lambda_m],$$

with $\lambda' < \lambda < \lambda''$ and $m \geq n$. Furthermore, this happens only if $\deg(h) > \deg(\varphi_n)$.

Proof. Proposition 6.3(ii) shows that D_{α_g} intersects the special fiber of $\mathcal{Z}_{g,0}$ at the intersection point z of the v'_g and v''_g-components. Since $\lambda' < \lambda_n < \lambda''$ by Remark 5.21, Corollary 5.4 then shows that the divisor D_{α_h} passes through z if and only if $\lambda' < \nu_K(\varphi_n(\alpha_h)) < \lambda''$. Now, by Corollary 4.4, $e(v_{n-1}/v_0) = \deg(\varphi_n)$, so $e(v'_g/v_0)$ and $e(v''_g/v_0)$ are both at least $\deg(\varphi_n)$. By [LL99, Lemma 5.1(a)], D_{α_h} can meet z only if $\deg(h) \geq e(v'_g/v_0) + e(v''_g/v_0) = 2\deg(\varphi_n)$. In this case, Corollary 4.12 shows that $v_h(\varphi_n) = \nu_K(\varphi_n(\alpha_h))$, so $\lambda' < v_h(\varphi_n) < \lambda''$.

Since $v_h(\varphi_n) > \lambda'$, the diskoid corresponding to v_h under the bijection in Proposition 4.6 is contained in $D(\varphi_n, \lambda')$, which is the diskoid corresponding to v'_g. By the same proposition, we have $v_h \succ v'_g$. [Rüt14, Proposition 4.35] now shows that

$$v_h = [v_0, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \lambda, \ldots, v_m(\varphi_m) = \lambda_m],$$

with $\lambda' < \lambda$ and $m \geq n$. Since $v_h(\varphi_n) < \lambda''$, we in fact have that $\lambda' < \lambda < \lambda''$. □

We use Lemma 10.6 to obtain a criterion for when it is possible for $\mathcal{W}_f$ to be different than the minimal regular model $\mathcal{Z}$ dominating all the $\mathcal{W}_{f_i}$. Note that $\mathcal{Z}$ is also the minimal normal model dominating the $\mathcal{W}_{f_i}$ by Lemma 5.5, since all the $\mathcal{W}_{f_i}$ include v_0.

Lemma 10.7. *Suppose $r = 2$, so that $f = \pi_K^b f_1 f_2$. Let $\mathcal{Z}$ be the minimal normal model of $\mathbb{P}^1_K$ dominating $\mathcal{W}_{f_1}$ and $\mathcal{W}_{f_2}$, and let $\mathcal{W}_f$ be defined as in Definition 10.4. Then $\mathcal{W}_f = \mathcal{Z}$ unless either*

- *f_1 is Eisenstein or linear, and f_2 is Eisenstein or linear, or,*
- *neither f_1 nor f_2 is linear, and $v_{f_1} = v_{f_2}$ (note that v_{f_i} is only defined when f_i is not linear).*

Proof. First, suppose that neither f_1 nor f_2 is Eisenstein or linear, so that the results of §6 apply. Let α_1, α_2 be roots of f_1, f_2 respectively. We show that if $v_{f_1} \neq v_{f_2}$, then D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}$, which means that $\mathcal{W}_f = \mathcal{Z}$. Without loss of generality, $v_{f_2} \not\preceq v_{f_1}$. Let $v_i := [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_i(\varphi_i) = \lambda_i]$ be maximal (under $\preceq$) such that $v_i \preceq v_{f_1}$ and $v_i \preceq v_{f_2}$. By [Rüt14, Proposition 4.35], we can write

$$v_{f_1} = [v_0, \ldots, v_i(\varphi_{i-1}) = \lambda_{i-1}, v_i(\varphi_i) = \lambda_i, v_{i+1}(\varphi_{i+1}) = \lambda_{i+1}, \ldots, v_n(\varphi_n) = \lambda_n]$$

and

$$v_{f_2} = [v_0, \ldots, v_i(\varphi_{i-1}) = \lambda_{i-1}, w_i(\varphi_i) = \mu_i, w_{i+1}(\psi_{i+1}) = \mu_{i+1}, \ldots, w_m(\psi_m) = \mu_m]$$

with $\lambda_i \leq \mu_i$. By Corollary 4.12, $\nu_K(\varphi_i(\alpha_1)) = \lambda_i$ and $\nu_K(\varphi_i(\alpha_2)) = \mu_i$.

First consider the case where $\lambda_i < \mu_i$. Proposition 5.12 shows that the valuation v_i is included in the minimal regular resolution of the v_{f_1}-model of $\mathbb{P}^1_K$. By Corollary 5.3 applied to this valuation, D_{α_1} and D_{α_2} do not meet on the v_i-model of $\mathbb{P}^1_K$. If $i < n$, then v_i is included in $\mathcal{Z}_{f_1,0}$, so they do not meet on $\mathcal{Z}_{f_1,0}$. Thus they do not meet on $\mathcal{Z}$. So $\mathcal{W}_f = \mathcal{Z}$. If $i < m$, then the same proof works using w_i (the ith truncation of v_{f_2}) and $\mathcal{Z}_{f_2,0}$ instead of v_i and $\mathcal{Z}_{f_1,0}$. If $i = m = n$, then define λ' and λ'' such that $[v_0, \ldots, v_n(\varphi_n) = \lambda']$ is the successor valuation v'_{f_1} to v_{f_1} and $[v_0, \ldots, v_n(\varphi_n) = \lambda'']$ is the precursor valuation v''_{f_1}. We define μ' and μ'' similarly using v_{f_2}. By Proposition 5.12 and the definition of $\mathcal{Z}_{f_1,0}$, the model $\mathcal{Z}_{f_1,0}$ includes v'_{f_1} and v''_{f_1}. Similarly, the model $\mathcal{Z}_{f_2,0}$ includes v'_{f_2} and v''_{f_2}. So $\mathcal{Z}$ includes v'_{f_1}, v''_{f_1}, v'_{f_2}, and v''_{f_2}. Since $\lambda_n = \lambda_i < \mu_i = \mu_n$, by Lemma 5.32, either $\lambda_n < \lambda'' \leq \mu_n$ or $\lambda_n \leq \mu' < \mu_n$. Corollary 5.3 shows that, in the first case, D_{α_1} and D_{α_2} do not meet on the v''_{f_1}-model of $\mathbb{P}^1_K$, and in the second case, D_{α_1} and D_{α_2} do not meet on the v'_{f_2}-model of $\mathbb{P}^1_K$. In all cases, they do not meet on $\mathcal{Z}$.

Now, suppose $\lambda_i = \mu_i$. By Lemma 4.3(iii), we have $\deg(\varphi_{i+1}) = \deg(\psi_{i+1})$ if $i < n$ and $\deg(f_1) = \deg(\psi_{i+1})$ if $i = n$. If $i < n$, then the fact that f_1 and f_2 are proper key polynomials shows that $\deg(f_1)$ and $\deg(f_2)$ are both greater than $\deg(\psi_{i+1})$, so Corollary 4.12 shows that $\nu_K(\psi_{i+1}(\alpha_1)) = v_{f_1}(\psi_{i+1})$ and $\nu_K(\psi_{i+1}(\alpha_2)) = v_{f_2}(\psi_{i+1})$. The maximality of v_i implies that we cannot have $w_{i+1} \preceq v_{f_1}, v_{f_2}$, and since $w_{i+1} \preceq v_{f_2}$, this implies that $w_{i+1} \not\preceq v_{f_1}$, or equivalently, $v_{f_1}(\psi_{i+1}) < v_{f_2}(\psi_{i+1}) = \mu_{i+1}$. Corollary 5.3 now shows that D_{α_1} and D_{α_2} do not meet on the v-model of $\mathbb{P}^1_K$ where $v = [v_0, \ldots, v_i(\varphi_i) = \lambda_i,\, w_{i+1}(\psi_{i+1}) = \lambda]$ for any λ satisfying $v_{f_1}(\psi_{i+1}) \leq \lambda \leq \mu_{i+1}$. Since $\mathcal{Z}_{f_2,0}$ includes at least one of these v, we conclude that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}_{f_2,0}$, and thus not on $\mathcal{Z}$. If $i = n$, then our assumption that $v_{f_2} \not\preceq v_{f_1}$ implies that $v_{f_2} \succ v_{f_1}$, so $m > n$. Since $\deg(f_1) = \deg(\psi_{n+1}) \leq \deg(\psi_m) < \deg(f_2)$, Lemma 10.6 with $g = f_2$ and $h = f_1$ shows that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}_{f_2,0}$, and thus not on $\mathcal{Z}$. Again, we conclude $\mathcal{W}_f = \mathcal{Z}$.

Lastly, suppose that f_1 is Eisenstein or linear, but f_2 is neither (by symmetry, this is the last case we need to consider). In both cases, $\mathcal{W}_{f_1,0}$ is the v_0-model of $\mathbb{P}^1_K$, so $\mathcal{Z} = \mathcal{Z}_{f_2,0}$. Recall that $v_{f_2} = [v_0, \ldots, w_m(\psi_m) = \mu_m]$. If f_1 is linear, then $\deg(f_1) \leq \deg(\psi_m)$, so Lemma 10.6 with $g = f_2$ and $h = f_1$ shows that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z} = \mathcal{Z}_{f_2,0}$, so $\mathcal{W}_f = \mathcal{Z}$. Now assume f_1 is Eisenstein of degree d. By Remark 5.31 we have $v_{f_1} = [v_0, v_1(x) = 1/d]$. Since f_2 is not linear, Lemma 4.3(i) implies $m \geq 1$. If $m \geq 2$, then Lemma 10.6 with $g = f_2$ and $h = f_1$ shows that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z} = \mathcal{Z}_{f_2,0}$. If $n = 1$, then $v'_{f_2} = [v_0,\, v_1(\psi_1) = \mu']$ and $v''_{f_2} = [v_0,\, v_1(\psi_1) = \mu'']$. Since f_2 is neither Eisenstein nor linear, Lemma 6.16 implies $v'_{f_2} \neq v_0$, so $\mu' > 0$. By Proposition 6.3(ii), D_{α_2} meets the special fiber of $\mathcal{Z}_{f_2,0}$ at the intersection z of the v'_{f_2} and v''_{f_2}-components, and by Corollary 5.4, the same is true of D_{α_1} only if $\mu' < 1/d < \mu''$. Since $\mu' > 0$, this means the denominator of μ', which is equal to $e(v'_{f_2}/v_0)$, is greater than d. Now [LL99, Lemma 5.1(a)] shows that no degree d point can specialize to z. Again, we conclude that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z} = \mathcal{Z}_{f_2,0}$. In both cases, we have $\mathcal{W}_f = \mathcal{Z}$. □

Proposition 10.8 below is the key to passing from the case of the conductor-discriminant inequality where f is irreducible to the general case. Parts (i), (ii), (iii), and (iv) of Proposition 10.8 correspond to the analogous parts of Lemma 2.7.

Proposition 10.8. *Suppose $r = 2$, so that $f = \pi_K^b f_1 f_2$. Let α_i be a root of f_i for $i \in \{1, 2\}$. Let $\mathcal{Z}$ be the minimal normal (equivalently, regular) model of $\mathbb{P}^1_K$ dominating $\mathcal{W}_{f_1}$ and $\mathcal{W}_{f_2}$, and let $N_{\mathcal{Z}}$ be the number of irreducible components of the special fiber of $\mathcal{Z}$. Then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2}$. Furthermore:*

- (i) *If exactly one of $\mathcal{W}_{f_1}$ or $\mathcal{W}_{f_2}$ is the v_0-model of $\mathbb{P}^1_K$, then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2} - 2$.*
- (ii) *If there exists $\gamma \in K$ such that $\nu_K(\alpha_1 - \gamma) = a/2$ with $a \geq 3$ odd, and f_2 linear with $\nu_K(\alpha_2 - \gamma) > a/2$, then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2} - 3$.*
- (iii) *If f_1 is non-Eisenstein with $\deg(f_1) \geq 3$ and f_2 is Eisenstein with $\deg(f_2) \geq 2$, then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2} - 3$.*
- (iv) *If neither $\mathcal{W}_{f_1}$ nor $\mathcal{W}_{f_2}$ is the v_0-model, and $v_{f_1} \neq v_{f_2}$, then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2} - 4$.*
- (v) *In general, if neither $\mathcal{W}_{f_1}$ nor $\mathcal{W}_{f_2}$ is the v_0-model, then $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2} - 3$.*

Proof. We will first show that in cases (i)-(iv), the divisors D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}$. If we are in the situation of part (i), Lemma 10.7 shows that $\mathcal{W}_f = \mathcal{Z}$, so $N_{\mathcal{W}_f} = N_{\mathcal{Z}}$. Since one of the f_i is neither linear nor Eisenstein, Lemma 2.7(i) shows that $\rho_{f_1,f_2} \geq 2$. This proves part (i).

If f_1 and f_2 are as in part (ii), then f_1 is neither Eisenstein nor linear and f_2 is linear. So we are in the situation of part (i), but this time Lemma 2.7(ii) shows that $\rho_{f_1,f_2} \geq 3$. Part (ii) follows.

If f_1 and f_2 are as in part (iii), then $v_{f_2} = [v_0,\, v_1(x) = 1/\deg(f_2)] \neq v_{f_1}$ (see Remark 5.31). Since $v_{f_1} \neq v_{f_2}$, Lemma 10.7 shows that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}$, so $\mathcal{W}_f = \mathcal{Z}$. By Lemma 2.7(iii), $\rho_{f_1,f_2} \geq 3$, proving part (iii).

If neither f_1 nor f_2 is linear or Eisenstein, then by Lemma 10.7, $v_{f_1} \neq v_{f_2}$ implies that D_{α_1} and D_{α_2} do not meet on $\mathcal{Z}$, so $\mathcal{W}_f = \mathcal{Z}$. Since Lemma 2.7(iv) shows that $\rho_{f_1,f_2} \geq 4$ in this case, this proves part (iv).

Now we will assume D_{α_1} and D_{α_2} meet on $\mathcal{Z}$. We prove $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \rho_{f_1,f_2}$ in each of these cases, and verify all remaining cases of part (v) when further neither $\mathcal{W}_{f_1}$ nor $\mathcal{W}_{f_2}$ is the v_0-model. By Lemma 10.7, up to symmetry, there are 4 possibilities:

(1) f_1 and f_2 are both linear,
(2) f_1 is Eisenstein of degree at least 2 and f_2 is linear,
(3) f_1 and f_2 are both Eisenstein of degree at least 2,
(4) or $v_{f_1} = v_{f_2}$, and neither f_1 nor f_2 is linear or Eisenstein.

Case (1): The model $\mathcal{Z}$ is the v_0-model of $\mathbb{P}^1_K$. By Corollary 5.3, the divisors D_{α_1} and D_{α_2} are separated on the minimal regular resolution $\mathcal{W}$ of the $\{w, v_0\}$-model of $\mathbb{P}^1_K$, where $w = [v_0,\, v_1(f_1) = \nu_K(\alpha_1 - \alpha_2)]$. Thus $\mathcal{W}_f$ is dominated by $\mathcal{W}$. Since $\nu_K(\alpha_1 - \alpha_2) \in \mathbb{Z}$, Corollary 5.16(i) shows that the model $\mathcal{W}$ includes exactly the valuations $[v_0,\, v_1(f_1) = \lambda]$ with $\lambda \in \{0, 1, \ldots, \nu_K(\alpha_1 - \alpha_2)\}$. So $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \nu_K(\alpha_1 - \alpha_2)$. Since $\rho_{f_1,f_2} = \nu_K(\alpha_1 - \alpha_2)$, case (1) is proved.

Case (2): Again, $\mathcal{Z}$ is the v_0-model of $\mathbb{P}^1_K$. Since $\nu_K(\alpha_2) \geq 1$ and $\nu_K(\alpha_1) < 1$, Proposition 5.2 shows that the divisors D_{α_1} and D_{α_2} do not meet on the w-model of $\mathbb{P}^1_K$, where $w = [v_0,\, v_1(x) = 1]$. So $\mathcal{W}_f$ includes exactly the valuations v_0 and w. Thus $N_{\mathcal{W}_f} - N_{\mathcal{Z}} = 1$. On the other hand, $\nu_K(\alpha_1 - \alpha_2) = \nu_K(\alpha_1) = 1/\deg(f_1)$ since f_1 is Eisenstein, so $\rho_{f_1,f_2} = \deg(f_1)/\deg(f_1) = 1$. Case (2) follows.

Case (3): Again, $\mathcal{Z}$ is the v_0-model of $\mathbb{P}^1_K$. Let d_1 and d_2 be the degrees of f_1 and f_2, respectively, and assume without loss of generality that $d_1 \geq d_2$. So $v_{f_i} = [v_0, v_1(x) = 1/d_i]$ for $i \in \{1, 2\}$, see Remark 5.31.

First assume that $d_1 > d_2$. By Corollary 5.3, D_{α_1} and D_{α_2} do not meet on the minimal regular resolution $\mathcal{W}$ of the $\{v_0, v_{f_2}\}$-model of $\mathbb{P}^1_K$. So $\mathcal{W}_f$ is dominated by $\mathcal{W}$, which includes exactly the valuations $[v_0, v_1(x) = \lambda]$, for $\lambda \in \{0, 1, 1/2, 1/3, \ldots, 1/d_2\}$, as can be computed by Proposition 5.12. So $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq d_2$. Also, $\rho_{f_1,f_2} = d_1 d_2 \nu_K(\alpha_1) = d_2$. So the desired inequality holds when $d_1 > d_2$.

Now assume $d_1 = d_2 =: d$. Then we define $\lambda_2 = \nu_K(f_1(\alpha_2))$. By Corollary 5.3, D_{α_1} and D_{α_2} do not meet on the minimal regular resolution $\mathcal{W}$ of the $\{v, v_0\}$-model of $\mathbb{P}^1_K$, where $v = [v_0,\, v_1(x) = 1/d,\, v_2(f_1(x)) = \lambda_2]$. So $\mathcal{W}_f$ is dominated by $\mathcal{W}$. By Proposition 5.12, the irreducible components of the special fiber of $\mathcal{W}$ other than v_0 are contained among the $v_{1,\lambda}$, $v_{2,\lambda}$, $w_{0,\lambda}$, and $w_{1,\lambda}$ in the notation of that proposition. Note that $\lambda_2 \in (1/d)\mathbb{Z} = \Gamma_{v_1}$ since α_2 has degree d over K. Applying Corollary 5.17(ii) to the $w_{0,\lambda}$ and $v_{1,\lambda}$, and Corollary 5.17(iii) to the $w_{1,\lambda}$ and $v_{2,\lambda}$, and throwing on an extra "1" for the v_0-component, we have

$$N_{\mathcal{W}} \leq \underbrace{1}_{v_0} + \underbrace{0 - 0 + d}_{w_{0,\lambda},\, v_{1,\lambda}} + \underbrace{d(\lambda_2 - 1)}_{w_{1,\lambda},\, v_{2,\lambda}} = 1 + d\lambda_2.$$

Since $\rho_{f_1,f_2} = d\nu_K(f_1(\alpha_2)) = d\lambda_2$ and $N_{\mathcal{Z}} = 1$, we have the desired inequality for case (3).

Case (4): First, observe that $\mathcal{Z} = \mathcal{Z}_{f_1,0} = \mathcal{Z}_{f_2,0}$. Write $v = v_{f_1} = v_{f_2} = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$. Let $\lambda_{n+1} = \nu_K(f_1(\alpha_2))$. Then Corollary 5.3 shows that D_{α_1} and D_{α_2} do not meet on the minimal regular resolution $\mathcal{W}$ of the $\{w, v_0\}$ model of $\mathbb{P}^1_K$, where $w = [v_0, \ldots, v_n(\varphi_n) = \lambda_n,\, v_{n+1}(f_1) = \lambda_{n+1}]$. So $\mathcal{W}$ dominates $\mathcal{W}_f$. By Proposition 5.12, the irreducible components of the special fiber of $\mathcal{W}$ that are not already included in $\mathcal{Z}$ correspond to v, as well as to the $w_{n,\lambda}$ and the $v_{n+1,\lambda}$ in the language of that proposition. Observe that $\lambda_{n+1} \in (1/\deg(f_2))\mathbb{Z}$, which is Γ_v by Corollary 4.4. Applying Corollary 5.17(iii), we have that the number of components in $\mathcal{W}$ but not $\mathcal{Z}$ is bounded above by

$$1 + \deg(f_2)(\lambda_{n+1} - e(v_n/v_{n-1})\lambda_n).$$

Since $e(v_n/v_{n-1}) \geq 2$ and Lemma 4.3(iv) and (vi) show that $\deg(f_2)\lambda_n \geq 2$, the quantity above is at most $\deg(f_2)\lambda_{n+1} - 3$. Since $\rho_{f_1,f_2} = \deg(f_2)\nu_K(f_1(\alpha_2)) = \deg(f_2)\lambda_{n+1}$, we have $N_{\mathcal{W}_f} - N_{\mathcal{Z}} \leq \deg(f_2)\lambda_{n+1} - 3$ as desired. This finishes case (4). $\square$

Remark 10.9. For future reference in the proof of Lemma 10.14, we record here that in case (1), the model $\mathcal{W}_f$ includes at most the valuations $[v_0,\, v_1(f_1) = \lambda]$ with $\lambda \in \{0, 1, \ldots, \nu_K(\alpha_1 - \alpha_2)\}$. In case (2), $\mathcal{W}_f$ includes exactly v_0 and $[v_0,\, v_1(x) = 1]$. In case (3), if $\deg(f_1) > \deg(f_2)$, then $\mathcal{W}_f$ includes at most the valuations $[v_0,\, v_1(x) = \lambda]$, where $\lambda \in \{0, 1, 1/2, \ldots, 1/\deg(f_2)\}$. If $\deg(f_1) = \deg(f_2)$, then $\mathcal{W}_f$ includes at most those valuations included in the minimal regular resolution of the v-model of $\mathbb{P}^1_K$, where $v = [v_0,\, v_1(x) = 1/d,\, v_2(f_1) = \lambda_2]$, where $d = \deg(f_1) = \deg(f_2)$ and $\lambda_2 = \nu_K(f_1(\alpha_2)) \in \Gamma_{v_1}$. By Proposition 5.12, the extra valuations included in this model that are not included when $\deg(f_1) > \deg(f_2)$ are of the form $[v_0,\, v_1(x) = 1/d,\, v_2(f_1) = \lambda]$ with $\lambda \leq \lambda_2$.

Remark 10.10. In the situation of Proposition 10.8, observe that no matter what, the horizontal parts of $\operatorname{div}_0(f_1)$ and $\operatorname{div}_0(f_2)$ meet at the specialization of $x = 0$ on the v_0-model of $\mathbb{P}^1_K$. So $\mathcal{W}_f$ is a regular model that includes more than just v_0. In fact, by Corollary 5.16(iv), it includes $[v_0,\, v_1(x) = 1]$. So if f_1 and f_2 are both either Eisenstein or linear, then $[v_0,\, v_1(x) = 1]$ is included in $\mathcal{W}_f$ but not in $\mathcal{Z}$. Furthermore, if some f_i is neither Eisenstein nor linear, then $[v_0,\, v_1(x) = 1]$ is already included in $\mathcal{W}_{f_i}$ (Lemma 6.16), and thus in $\mathcal{Z}$.

10.4. **Obtaining $\mathcal{W}_f$ from the $\mathcal{W}_{f_i}$ — many irreducible polynomial factors.** In this subsection, we extend Proposition 10.8 to arbitrary r, and use it to place an upper bound on the number of irreducible components on the special fiber of $\mathcal{W}_f$. The first two terms of the left-hand side of the inequality in Corollary 10.11 below are the main terms, but we also need to record some "cost savings" in order to get a bound that will work for some of the "edge cases" of the conductor-discriminant inequality.

Corollary 10.11. *Let $f = \pi_K^b f_1 \cdots f_r$ as in this section and assume $r \geq 2$. Let $\mathcal{Z}$ be the minimal normal model of $\mathbb{P}^1_K$ dominating all the $\mathcal{W}_{f_i}$, and let $\mathcal{W}_f$ be the minimal regular model of $\mathbb{P}^1_K$ dominating $\mathcal{Z}$ and separating the horizontal parts of the $\operatorname{div}_0(f_i)$. Define $N_{\mathcal{Z}}$,*

$N_{\mathcal{W}_{f_i}}$, *and* $N_{\mathcal{W}_f}$ *to be the number of irreducible components on the special fiber of* $\mathcal{Z}$, $\mathcal{W}_{f_i}$, *and* $\mathcal{W}_f$, *respectively. Let* s *be the number of* f_i *that are neither linear nor Eisenstein. Then*

$$N_{\mathcal{W}_f} - 1 \leq \sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1) + \sum_{1\leq i<j\leq r} \rho_{f_i,f_j} - \left(\binom{r}{2} + 2\binom{s}{2} + s(r-s) + (s-1)\right) - \epsilon,$$

where $\epsilon = 1$ *in the following cases:*

- $s \geq 2$,
- $s = 1$, *there exists* $\gamma \in K$ *along with* f_i, f_j *with roots* α_i, α_j *such that* $\nu_K(\alpha_i - \gamma) = a/2$ *with* $a \geq 3$ *odd, and* f_j *linear with* $\nu_K(\alpha_j - \gamma) > a/2$ *(cf. Proposition 10.8(ii)),*
- $s = 1$, *there exists a non-Eisenstein* f_i *such that* $\deg(f_i) \geq 3$, *and there exists an Eisenstein, non-linear* f_j *(cf. Proposition 10.8(iii)),*

Otherwise, $\epsilon = 0$.

Proof. Number the f_i so that $f_1, \ldots, f_s$ are neither Eisenstein nor linear. Since all $\mathcal{W}_{f_i}$ include v_0, Lemma 5.5 implies that $\mathcal{Z}$ is regular and $N_{\mathcal{Z}} - 1 \leq \sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1)$. In fact, since all $\mathcal{W}_{f_i}$ for $1 \leq i \leq s$ include $[v_0,\, v_1(x) = 1]$ (Lemma 6.16), we even have that $N_{\mathcal{Z}} - 1 \leq \sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1) - (s-1)$, since we need only count the component corresponding to $[v_0, v_1(x) = 1]$ once total, and not for each f_i with $1 \leq i \leq s$.

Suppose $1 \leq i < j \leq r$. By Proposition 10.8, separating the horizontal parts of $\mathrm{div}_0(f_i)$ and $\mathrm{div}_0(f_j)$ on any model that dominates both $\mathcal{W}_{f_i}$ and $\mathcal{W}_{f_j}$ requires at most an *additional* ρ_{f_i,f_j} closed point blowups. Furthermore, Proposition 10.8(i) shows that if f_i is neither Eisenstein nor linear but f_j is, then only $\rho_{f_i,f_j} - 2$ additional closed point blowups are required. If neither f_i nor f_j is Eisenstein or linear, then Proposition 10.8(v) shows that only $\rho_{f_i,f_j} - 3$ additional closed point blowups are required. Also observe that, by Remark 10.10, if both f_i and f_j are Eisenstein/linear, then $[v_0,\, v_1(x) = 1]$ corresponds to the exceptional divisor of one of the blowups necessary to separate $\mathrm{div}_0(f_i)$ and $\mathrm{div}_0(f_j)$. This irreducible component is counted once for each such pair of f_i and f_j. Since there are $r - s$ such f_i, it is *overcounted* $\binom{r-s}{2} - 1$ times. If $s \geq 1$, then $[v_0,\, v_1(x) = 1]$ is already included in $\mathcal{W}_{f_1}$ by Remark 10.10, and thus the corresponding irreducible component is actually overcounted $\binom{r-s}{2}$ times. Putting all this together, we obtain that

$$N_{\mathcal{W}_f} - 1 \leq \sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1) + \sum_{1\leq i<j\leq r} \rho_{f_i,f_j} - 2s(r-s) - 3\binom{s}{2} - (s-1) - \binom{r-s}{2}. \tag{10.12}$$

Since $\binom{r}{2} = \binom{r-s}{2} + \binom{s}{2} + s(r-s)$, the right-hand side of (10.12) equals

$$\sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1) + \sum_{1\leq i<j\leq r} \rho_{f_i,f_j} - \left(\binom{r}{2} + 2\binom{s}{2} + s(r-s) + (s-1)\right). \tag{10.13}$$

It remains to show that, when $\epsilon = 1$, the expression in (10.13) overcounts $N_{\mathcal{W}_f} - 1$ by at least 1. If $s = 1$ and $\epsilon = 1$, let f_i and f_j be as in the definition of ϵ. Then by Proposition 10.8(ii) or (iii), at most $\rho_{f_i,f_j} - 3$ additional closed point blowups are needed to separate $\mathrm{div}_0(f_1)$ and $\mathrm{div}_0(f_2)$, as opposed to the $\rho_{f_i,f_j} - 2$ counted above, leading to the overcount. Now assume $s \geq 2$ and $\epsilon = 1$. Either $v_{f_1} \neq v_{f_2}$ or $v_{f_1} = v_{f_2}$. If $v_{f_1} \neq v_{f_2}$, then Proposition 10.8(v) shows that at most $\rho_{f_1,f_2} - 4$ additional closed point blowups are needed to separate $\mathrm{div}_0(f_1)$ and $\mathrm{div}_0(f_2)$, as opposed to the $\rho_{f_1,f_2} - 3$ counted above, leading to the overcount. If $v_{f_1} = v_{f_2}$, then Remark 10.3(i) and (ii) show that $\mathcal{W}_{f_1} = \mathcal{W}_{f_2}$ includes some valuation other than v_0 and $[v_0,\, v_1(x) = 1]$. We have counted this valuation twice (once in $N_{\mathcal{W}_{f_1}}$ and once in $N_{\mathcal{W}_{f_2}}$), leading to the overcount. ☐

10.5. **Obtaining $\mathcal{Y}_f$ from $\mathcal{W}_f$.** Recall from Definition 10.4 that the model $\mathcal{V}_f$ of $\mathbb{P}^1_K$ is the minimal modification of $\mathcal{W}_f$ that separates the odd horizontal components of $\operatorname{div}_0(f)$ from the odd vertical ones, and that $\mathcal{Y}_f$ is the minimal modification of $\mathcal{V}_f$ that further separates the odd vertical components from each other.

10.5.1. *Building $\mathcal{V}_f$.* The first two lemmas of this section are necessary for some exceptional edge cases of the conductor-discriminant inequality. They can be skipped on a first reading. The main results are Propositions 10.16 and 10.17.

Lemma 10.14. *Suppose that $f = f_1 f_2$ with each f_i monic, irreducible, and either Eisenstein or degree 1. Then $\operatorname{div}(f)$ has no odd vertical components on $\mathcal{W}_f$. So $\mathcal{V}_f = \mathcal{W}_f$.*

Proof. Assume without loss of generality that $\deg(f_1) \geq \deg(f_2)$. First suppose that $\deg(f_1) = 1$, thus $\deg(f_2) = 1$. Write $f_i = x - a_i$ for $i = 1, 2$. By Remark 10.9, the model $\mathcal{W}_f$ includes at most the valuations $v_\lambda := [v_0,\, v_1(f_1) = \lambda]$ for $\lambda \in \{0, 1, \ldots, \nu_K(a_1 - a_2)\}$. One checks that $v_\lambda(f_1) = v_\lambda(f_2) = \lambda$ for all λ, and therefore $v_\lambda(f) = v_\lambda(f_1) + v_\lambda(f_2) = 2\lambda$ which means the order of f is even on each vertical component, proving the lemma in this case.

Now, assume that f_1 is Eisenstein of degree > 1. The roots of f_1 have valuation $1/\deg(f_1)$. If $\deg(f_2) = 1$, then by Remark 10.9, $\mathcal{W}_f$ includes exactly the valuations v_0 and $v := [v_0,\, v_1(x) = 1]$. Since $f_2 = x - a$ with $\nu_K(a) \geq 1$, we see that $v_0(f_1) = v_0(f_2) = 0$, and $v(f_1) = v(f_2) = 1$, which as before imply that $v_0(f) = 0$ and $v(f) = 2$. So the order of f is even on each vertical component, again proving the lemma.

If $\deg(f_1) > \deg(f_2) > 1$ and f_1 and f_2 are both Eisenstein, then by Remark 10.9, $\mathcal{W}_f$ includes at most the valuations $v_\lambda := [v_0,\, v_1(x) = \lambda]$, for $\lambda \in \{0, 1, 1/2, 1/3, \ldots, 1/\deg(f_2)\}$. One checks that $v_0(f_1) = v_0(f_2) = 0$, and $v_\lambda(f_1) = v_\lambda(f_2) = 1$ for all $\lambda > 0$, which would in turn imply that $v_0(f) = 0$ and $v_\lambda(f) = 2$. Again, the order of f is even on each vertical component, proving the lemma.

If $\deg(f_1) = \deg(f_2) = d$ and f_1 and f_2 are both Eisenstein, then again by Remark 10.9, $\mathcal{W}_f$ includes at most the valuations in the previous paragraph, as well as other valuations of the form $w_\lambda := [v_0,\, v_1(x) = 1/d,\, v_2(f_1) = \lambda]$, for various values of $\lambda \leq \nu_K(f_1(\alpha_2))$. Now, since $\deg(f_1 - f_2) < \deg(f_1)$ and f_2 is a key polynomial over $[v_0,\, v_1(x) = 1/d]$, Corollary 4.12 shows that $v_1(f_1 - f_2) = \nu_K((f_1 - f_2)(\alpha_2)) = \nu_K(f_1(\alpha_2))$. Thus, for each $\lambda \leq \nu_K(f_1(\alpha_2))$, [Rüt14, Theorem 4.33] shows that w_λ remains the same when f_1 is replaced by f_2. In particular, $w_\lambda(f_2) = \lambda = w_\lambda(f_1)$ and therefore $w_\lambda(f) = w_\lambda(f_1) + w_\lambda(f_2) = 2\lambda \in 2\Gamma_{w_\lambda}$ for all $\lambda \leq \nu_K(f_1(\alpha_2))$. So the order of $f = f_1 f_2$ is even on each vertical component corresponding to a w_λ, proving the lemma in this case. ☐

Lemma 10.15. *Let $f = \pi_K^b f_1 \cdots f_r$ with $r \geq 2$ as in this section, let α_i be a root of f_i for all i, and order the f_i so that $\nu_K(\alpha_i)$ is non-decreasing in i. Let $d \in \mathbb{N}$ be minimal such that $1/d \leq \nu_K(\alpha_2)$.*

(i) *The model $\mathcal{W}_f$ includes the valuation $[v_0,\, v_1(x) = \lambda]$ for all $\lambda \in \{0, 1, 1/2, 1/3, \ldots, 1/d\}$.*
(ii) *Suppose f_i is Eisenstein of degree c and $\mathcal{W}_f$ does not include $[v_0,\, v_1(x) = 1/c]$. If $q \in \mathbb{N}$ is maximal such that $\mathcal{W}_f$ includes $v := [v_0,\, v_1(x) = 1/q]$, then $q < c$ and D_{α_i} meets the intersection z of the v_0 and v-components of the special fiber of $\mathcal{W}_f$.*

(iii) *In the situation of part (ii), If* $b = 0$ *and* v *corresponds to an odd component for* $\mathrm{div}(f)$*, then after blowing up the point* z*, the divisor* D_{α_i} *from part (ii) intersects only even vertical components of* $\mathrm{div}(f)$*.*

Proof. We start with part (i). Since $\mathcal{W}_f$ is regular and contains v_0, and since $1 > 1/2 > \cdots > 1/d$ is a shortest 1-path from 1 to $1/d$, it suffices by Proposition 5.12 to show that $\mathcal{W}_f$ includes $[v_0, v_1(x) = 1/d]$. Now, $\mathcal{W}_f$ is includes more than just v_0, because otherwise all the D_{α_i} would meet on the special fiber. By Corollary 5.16(iii), $\mathcal{W}_f$ includes a valuation of the form $[v_0, v_1(x) = 1/q]$ for $q \in \mathbb{N}$. If q is maximal with this property, then the v_0-component and the $[v_0, v_1(x) = 1/q]$-component intersect at the point z on $\mathcal{W}_f$ by Remark 5.15, and the induced map of models from $\mathcal{W}_f$ to the $\{v_0, [v_0, v_1(x) = 1/q]\}$-model is an isomorphism in a neighbourhood of z.

If $q \geq d$, then Corollary 5.16(iv) shows that $\mathcal{W}_f$ includes $[v_0, v_1(x) = 1/d]$. Now assume $q \leq d-1$. Since d is minimal with the property $1/d \leq \nu_K(\alpha_2)$, we have $\nu_K(\alpha_2) < 1/(d-1)$ and since $\nu_K(\alpha_1) \leq \nu_K(\alpha_2)$ by convention, it follows that $0 < \nu_K(\alpha_i) < 1/(d-1) \leq 1/q$ for $i \in \{1, 2\}$. By Corollary 5.4, D_{α_1} and D_{α_2} intersect at z in the $\{v_0, [v_0, v_1(x) = 1/q]\}$-model, and therefore also in $\mathcal{W}_f$. But this contradicts the definition of $\mathcal{W}_f$. Thus $\mathcal{W}_f$ must include $[v_0, v_1(x) = 1/d]$, proving part (i).

By Corollary 5.16(iv), $\mathcal{W}_f$ not including $[v_0, v_1(x) = 1/c]$ means that $q < c$. Then $0 < 1/c < 1/q$. If α_i is as in part (ii), then $0 < \nu_K(\alpha_i) < 1/q$. By Corollary 5.4, D_{α_i} intersects the special fiber of the $\{v_0, [v_0, v_1(x) = 1/q]\}$-model and therefore also $\mathcal{W}_f$ at z, proving part (ii).

To prove part (iii), observe that if $b = 0$, then the v_0-component is even since $v_0(f) = 0$. The v-component is odd by assumption. We will show that the exceptional divisor E of the blowup $\mathcal{W}'_f \to \mathcal{W}_f$ at z is an even component of $\mathrm{div}(f)$, and that D_{α_1} either intersects a unique component of the special fiber (namely E), or meets the special fiber at the intersection of the two even components (the v_0-component and E). The horizontal component of $\mathrm{div}(f)$ intersecting z is D_{α_1}, and D_{α_1} appears with multiplicity 1 in $\mathrm{div}(f)$. By the definition of $\mathcal{W}_f$, no other components of $\mathrm{div}(f)$ intersect z. Furthermore, the definition of $\mathcal{W}_f$ implies that D_{α_i} is regular on $\mathcal{W}_f$. So Lemma 2.11 and Remark 2.12 apply to show that E is an even component of $\mathrm{div}(f)$. Since E intersects the v_0 and v-components and since $\mathcal{W}'_f$ is regular, [OW18, Lemma 7.4] and the definition of 1-path show that $\nu_E = [v_0, v_1(x) = \lambda]$, with $1/q > \lambda > 0$ a 1-path. This determines λ, which must be $1/(q+1)$. If $c = \deg(f) = q+1$, then since E has multiplicity $q+1$ and α_i has K-degree $q+1$, [LL99, Lemma 5.1(a)] shows that D_{α_1} meets only the component E on the special fiber of $\mathcal{W}'_f$, which is an even component of $\mathrm{div}(f)$. If $q+1 < c$, then arguing as in parts (i) and (ii) using Corollary 5.4, we see that D_{α_1} meets the intersection of E and the v_0-component in $\mathcal{W}'_f$, both of which are even components of $\mathrm{div}(f)$. Since $q < c$ by (ii), this finishes the proof of part (iii). □

Proposition 10.16. *Let* $f = \pi_K^b f_1 \cdots f_r$ *be as in this section with* $r \geq 2$*, let* s *be the number of* f_i *that are neither linear nor Eisenstein, and let* c *be the largest degree of any Eisenstein* f_i *(with* $c = 1$ *if no* f_i *is Eisenstein). Let* $\tau = 1$ *if* $b = 0$*,* $r = 2$*, and* f_1 *and* f_2 *satisfy one of the properties in Lemma 5.33 (up to switching the order of* f_1 *and* f_2*). Let* $\tau = 0$ *otherwise.*

Then obtaining $\mathcal{V}_f$ *from* $\mathcal{W}_f$ *requires at most* $r + s + b(c-1) - \tau$ *closed point blowups.*

Proof. Let α_i be a root of f_i for each f. Assume first that f_i is neither linear nor Eisenstein. Since $\mathcal{W}_f$ dominates $\mathcal{W}_{f_i} = \mathcal{Z}_{f_i,0}$, we have that $\mathcal{Z}_{f_i,0}$ is a contraction of $\mathcal{W}_f$. Recall from

Definition 6.1 that $\mathcal{Y}_{f_i,0} \to Z_{f_i,0}$ is a closed point blowup, and $\mathcal{Y}_{f_i,0}$ includes v_{f_i}. Base changing to $\mathcal{W}_f$, there is a closed point blowup $\mathcal{W}'_f \to \mathcal{W}_f$ such that $\mathcal{W}'_f$ includes v_{f_i}. By Lemma 9.3 (with $g = f$ and $h = f_i$ in that lemma), only one further closed point blowup is required so that D_{α_i} does not meet an odd component of $\mathrm{div}(f)$, meaning that we need at most 2 closed point blowups per f_i in total. Furthermore, if $\tau = 1$, then $f_i = f_1$ and f_2 is either linear or Eisenstein. Since f_2 is Eisenstein or linear implies that $\mathcal{W}_{f_2}$ is the v_0-model, by Lemma 10.7, we have $\mathcal{W}_f = \mathcal{W}_{f_1} = \mathcal{Z}_{f_1,0}$. This further means $\mathcal{W}'_f = \mathcal{Y}_{f_1,0}$ is the minimal regular resolution of the v_{f_1}-model of $\mathbb{P}^1_K$ including v_0. By Lemma 5.27(ii), the component that D_{α_i} intersects on $\mathcal{W}'_f$ is the v_{f_1}-component, which is an even component for $\mathrm{div}(f)$ by Lemma 5.33. So in this case, only one blowup total is needed to separate D_{α_i} from the odd components of $\mathrm{div}(f)$.

If f_i is linear, then Lemma 9.3 shows that at most one closed point blowup is necessary to separate D_{α_i} from the odd components of $\mathrm{div}(f)$.

Now, arrange the Eisenstein f_i so that their roots have non-decreasing valuation (i.e., in non-increasing order of degree). Let d' be the degree of the *second* polynomial in this sequence (or $d' = 1$ if there is only one Eisenstein polynomial). The roots of this polynomial have K-valuation $1/d'$. If d is as in Lemma 10.15, then $d' \leq d$, so Lemma 10.15(i) shows that $\mathcal{W}_f$ includes the components $v_\lambda := [v_0,\, v_1(x) = \lambda]$ for $\lambda \in \{0, 1, 1/2, 1/3, \ldots, 1/d'\}$.

For an Eisenstein polynomial f_i, we have $v_{f_i} = v_{1/\deg(f_i)}$, see Remark 5.31. So v_{f_i} is already included in $\mathcal{W}_f$ for all but possibly one Eisenstein factor f_i. For each of these f_i for which v_{f_i} is included, Lemma 9.3 shows that at most one point blowup is required to ensure that D_{α_i} intersects only even vertical components of $\mathrm{div}(f)$. For the last f_i with maximal degree c, if v_{f_i} is not included in $\mathcal{W}_f$, then Lemma 10.15(ii) and (iii) shows that if $b = 0$, then at most one blowup is required to separate D_{α_i} from the odd components of $\mathrm{div}(f)$.

If $b = 1$, then we consider the minimal regular model $\mathcal{W}''_f$ of $\mathbb{P}^1_K$ dominating $\mathcal{W}_f$ and including $v_c = [v_0,\, v_1(x) = 1/c]$. By Proposition 5.12, the minimal regular resolution of the v_c-model of $\mathbb{P}^1_K$ includes exactly the valuations v_0 and v_λ for $\lambda \in \{1, 1/2, \ldots, 1/c\}$. For each Eisenstein f_i, the valuation v_{f_i} is among these. By Lemma 5.5, $\mathcal{W}''_f$ includes exactly the union of these v_λ and the valuations already included in $\mathcal{W}_f$. Since $\mathcal{W}_f$ already includes $[v_0,\, v_1(x) = 1]$, we have that $\mathcal{W}''_f$ includes at most $c - 1$ vauations not already included in $\mathcal{W}_f$. Since every v_{f_i} is included for f_i Eisenstein, Lemma 9.3 shows that there is at most one more closed point blowup required for each Eisenstein f_i to ensure that D_{α_i} intersects only even vertical components of $\mathrm{div}(f)$.

So overall, we need at most $2s$ blowups for the f_i that are neither Eisenstein nor linear, 1 blowup each for the f_i that are Eisenstein or linear, and another $c - 1$ blowups if $b = 1$. Furthermore, if $\tau = 1$ (and thus $b = 0$), then one fewer blowup is needed. This totals to $2s + (r - s) - \tau = r + s - \tau$ if $b = 0$, and $2s + (r - s) + c - 1 = r + s + c - 1 - \tau$ if $b = 1$. □

Proposition 10.17. *Let $f = \pi_K^b f_1 \cdots f_r$ be as in this section with $r \geq 2$, let s be the number of f_i that are neither linear nor Eisenstein, and let c be the largest degree of any Eisenstein f_i (with $c = 1$ if no f_i is Eisenstein). Let $N_{\mathcal{W}_{f_i}}$ and $N_{\mathcal{V}_f}$ be the number of irreducible components of the special fiber of $\mathcal{W}_{f_i}$ and $\mathcal{V}_f$, respectively. Let τ be defined as in Proposition 10.16. Then*

$$N_{\mathcal{V}_f} - 1 \leq \sum_{i=1}^{r} (N_{\mathcal{W}_{f_i}} - 1) + \sum_{1 \leq i < j \leq r} \rho_{f_i, f_j} + 2 + bc - s - \tau,$$

with equality holding only if $r = 2$, $s = 1$, $b = 0$, and $\epsilon = 0$ as defined in Corollary 10.11.

Proof. If $r = 2$, $s = 0$, and $b = 0$, then $\tau = 0$. Lemma 10.14 shows that $\mathcal{W}_f = \mathcal{V}_f$, so the proposition follows, with a strict inequality, by Corollary 10.11.

In general, by Corollary 10.11 and Proposition 10.16, it suffices to prove that

$$(r + s + b(c-1) - \tau) - \left(\binom{r}{2} + 2\binom{s}{2} + s(r-s) + (s-1) + \epsilon\right) \leq 2 + bc - s - \tau,$$

with equality holding only when mentioned in the proposition. This is equivalent to

$$\binom{r}{2} + 2\binom{s}{2} + s(r-s) + \epsilon + b \geq r + s - 1.$$

with equality holding only when mentioned in the proposition. If $r \geq 3$, then $\binom{r}{2} \geq r$ and one checks that the inequality holds strictly. If $r = 2$, the inequality becomes $2\binom{s}{2} + s(1 - s) + \epsilon + b \geq 0$. Since $s \leq 2$, and $\epsilon = 1$ whenever $s = 2$, one again checks that the inequality holds, with equality only when $s = 1$ and $b = \epsilon = 0$, or when $s = b = 0$. But we already showed strict inequality in the case $s = b = 0$ at the beginning of the proof. □

10.5.2. *Building $\mathcal{Y}_f$.* By the construction of $\mathcal{V}_f$, no horizontal component of $\mathrm{div}_0(f)$ meets an odd vertical component of $\mathrm{div}(f)$. To construct $\mathcal{Y}_f$, we blow up the intersections of each pair of odd vertical components of $\mathrm{div}(f)$. By Lemma 2.11 and Remark 2.12, the exceptional divisor of such a blowup is an even component of $\mathrm{div}(f)$, and so no two odd vertical components of $\mathrm{div}(f)$ meet on $\mathcal{Y}_f$. If $N_{\mathcal{Y}_{f,\mathrm{even}}}$ is the number of even vertical components of $\mathrm{div}(f)$ on $\mathcal{Y}_f$, then it is a consequence of Lemma 9.2 that $N_{\mathcal{Y}_{f,\mathrm{even}}} \leq N_{\mathcal{V}_f} - 1$ whenever $\mathrm{div}(f)$ has an odd vertical component on $\mathcal{V}_f$. The following corollary, bounding $N_{\mathcal{Y}_f,\mathrm{even}}$, now follows relatively easily from Proposition 10.17.

Corollary 10.18. *Let $f = \pi_K^b f_1 \cdots f_r$ be as in this section with $r \geq 2$, let s be the number of f_i that are neither linear nor Eisenstein, and let c be the largest degree of any Eisenstein f_i (with $c = 1$ if no f_i is Eisenstein). Let $N_{\mathcal{W}_{f_i}}$ be the number of irreducible components of the special fiber of $\mathcal{W}_{f_i}$, and let $N_{\mathcal{Y}_f,\mathrm{even}}$ be the number of even irreducible vertical components of* $\mathrm{div}(f)$ *on $\mathcal{Y}_f$. Let τ be as in Proposition 10.16. Then*

$$N_{\mathcal{Y}_f,\mathrm{even}} - 1 \leq \sum_{i=1}^{r}(N_{\mathcal{W}_{f_i}} - 1) + \sum_{1 \leq i < j \leq r} \rho_{f_i,f_j} + 1 + bc - s - \tau,$$

with equality only if $r = 2$, $s = 1$, $b = 0$, and $\epsilon = 0$ in the language of Corollary 10.11.

Proof. If $\mathrm{div}(f)$ has an odd component on $\mathcal{V}_f$, the proposition follows from Proposition 10.17 and Lemma 9.2. If not, then by Lemma 9.2, $\mathcal{W}_f = \mathcal{V}_f$ and $N_{\mathcal{Y}_{f,\mathrm{even}}} = N_{\mathcal{V}_f} = N_{\mathcal{W}_f}$. From Corollary 10.11, it suffices to prove

$$\binom{r}{2} + 2\binom{s}{2} + s(r-s) + (s-1) + \epsilon + bc > s - 1 + \tau, \tag{10.19}$$

where $\epsilon \in \{0, 1\}$ is as defined in Corollary 10.11. From the definition of τ, if $\tau = 1$ then $s = r - s = 1$. Otherwise $\tau = 0$. Since $\binom{r}{2} \geq 1$, the inequality (10.19) holds. □

10.6. **Proof of the conductor-discriminant inequality.** We are finally ready to prove the conductor-discriminant inequality. For Theorem 10.20, we maintain our assumption that all roots of f have positive valuation.

Theorem 10.20. *Let $f = \pi_K^b f_1 \cdots f_r$, where $r \geq 2$, the $f_i \in \mathcal{O}_K[x]$ are distinct monic irreducible polynomials whose roots have positive valuation, and $b \in \{0, 1\}$. On the model $\mathcal{Y}_f$ of $\mathbb{P}^1_K$ from Definition 10.4, the odd part of $\operatorname{div}_0(f)$ is a disjoint union of regular irreducible components. Furthermore, $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y}_f,\mathrm{even}} - 1)$, so $\mathcal{Y}_f$ satisfies inequality (9.1).*

That is, the conductor-discriminant inequality holds for all f satisfying Assumption 2.1 with $r \geq 2$ and all roots of positive valuation.

Proof. The first assertion follows from the construction of $\mathcal{Y}_f$. We now place an upper bound on $N_{\mathcal{Y}_f,\mathrm{even}}$.

By Corollary 10.18, $N_{\mathcal{Y}_f,\mathrm{even}}$ satisfies

$$N_{\mathcal{Y}_f,\mathrm{even}} - 1 \leq \sum_{i=1}^{r} (N_{\mathcal{W}_{f_i}} - 1) + \sum_{1 \leq i < j \leq r} \rho_{f_i,f_j} + 1 + bc - s - \tau, \tag{10.21}$$

where s is the number of non-linear, non-Eisenstein f_i, where c is the largest degree of an Eisenstein f_i (with $c = 1$ if no f_i is Eisenstein), and where τ is as in Proposition 10.16 (recall that $\tau \in \{0, 1\}$). Equality holds only when $r = 2$, $s = 1$, $b = 0$, and $\epsilon = 0$ in the language of Corollary 10.11.

On the other hand, Definition 10.2, Theorem 8.1, along with Remark 10.3(iii), shows that $\operatorname{db}_K(f_i) \geq 2(N_{\mathcal{W}_{f_i}} - 1)$ for all i, except possibly for certain non-Eisenstein, non-linear f_i, for which $\operatorname{db}_K(f_i) \geq 2(N_{\mathcal{W}_{f_i}} - 2)$. Let t be the number of f_i corresponding to the exceptional cases mentioned in Remark 10.3(iii). Then $t \leq s$. By (2.6), we have

$$\operatorname{db}_K(f) \geq 2 \sum_i (N_{\mathcal{W}_{f_i}} - 1) - 2t + \sum_{1 \leq i < j \leq r} 2\rho_{f_i,f_j} + 2b(\deg(f) - 1). \tag{10.22}$$

Combining (10.21) with (10.22), and noting that $c \leq \deg(f) - 1$ since $r \geq 2$, we obtain that $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y}_f,\mathrm{even}} - 2 + s - t + \tau)$, with $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y}_f,\mathrm{even}} - 1 + s - t + \tau)$ unless $r = 2$, $s = 1$, and $b = \epsilon = 0$. Since $t \leq s$ and $\tau \geq 0$, we have $\operatorname{db}_K(f) \geq 2(N_{\mathcal{Y}_f,\mathrm{even}} - 1)$ unless $r = 2$, $s = t = 1$, and $b = \epsilon = 0$.

We claim that if $r = 2$, $s = t = 1$, and $b = \epsilon = 0$, then $\tau = 1$, which finishes the proof. Since $r = 2$ and $s = 1$, assume without loss of generality that $f = f_1 f_2$ where f_1 is neither Eisenstein nor linear and f_2 is Eisenstein or linear. Since $t = 1$, by Remark 10.3(iii) either $\nu_K(\alpha_1) = 2/c$ with $c \geq 3$ odd, or $\deg(f_1) = 2\deg(\varphi_n)$, where $v_{f_1} = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$. Going through the conditions for $\tau = 1$ when f_2 is Eisenstein non-linear (Lemma 5.33(iii)) and linear (Lemma 5.33(i) and (ii)), we have that $\tau = 1$ unless either

- f_2 is nonlinear and $\deg(f_1) \geq 3$,
- or there exists $\gamma \in K$ such that $\nu_K(\alpha_1 - \gamma) = a/2$ with $a \geq 3$ odd, and f_2 linear with $\nu_K(\alpha_2 - \gamma) > a/2$.

But since $\epsilon = 0$ and $s = 1$, the definition of ϵ in Corollary 10.11 shows that neither of these cases can occur. ∎

Proof of Theorem 1.1. By Theorems 9.4 and 10.20, the conductor-discriminant inequality holds for all $f \in \mathcal{O}_K[x]$ satisfying Assumption 2.1, all of whose roots have positive valuation.

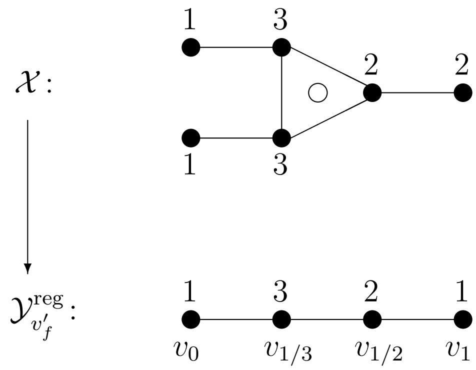


FIGURE 4. The modified dual graph of the models $\mathcal{X}$ and $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ in Example A.1. Components are labeled with their multiplicities. Components of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ are also labeled with their corresponding valuations.

Theorem 1.1 now follows by applying Corollary 3.3, Proposition 2.2, and Proposition 1.5 in sequence. □

APPENDIX A. EXAMPLES

We revisit Examples 1.9 and 1.10 in the context of Mac Lane valuations.

Example A.1 (Example 1.9, revisited)**.** Let X be the hyperelliptic curve with affine equation $y^2 = f(x)$, with $f(x) = x^8 - \pi_K^3$. Then f is irreducible, and f is a proper key polynomial over $v_f := [v_0,\, v_1(x) = 3/8]$. As in Example 6.8, the successor valuation is $[v_0,\, v_1(x) = 1/3]$, and the model $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ contains exactly the valuations $v_\lambda := [v_0,\, v_1(x) = \lambda]$ for $\lambda \in \{0, 1/3, 1/2, 1\}$. The horizontal part of $\mathrm{div}_0(f)$ is regular on this model, and the corresponding irreudicible components of the special fiber have multiplicities 0, 3, 2, and 1 respectively. Furthermore, we have

$$v_0(f) = 1, v_{1/3}(f) = 8/3, \text{ and } v_{1/2}(f) = v_1(f) = 3.$$

Thus the order of f is even on all components other than v_1. Since the roots of f have valuation 3/8, the horizontal part of $\mathrm{div}_0(f)$ meets the intersection of the $v_{1/3}$- and $v_{1/2}$-components (Corollary 5.4). We conclude that the odd irreducible components of $\mathrm{div}_0(f)$ are regular and disjoint, and the normalization $\mathcal{X}$ of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ in $K(X)$ is regular. In fact, $\mathcal{X}$ does not have normal crossings: three irreducible components at the special fiber meet at the same point. We draw a "modified" dual graph of $\mathcal{X}$ in Figure 4, where the triangle with a dot inside means that all three corresponding components meet at the same point. We also show how the components map down to $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ (compare to Figure 1 in Example 1.9).

Notice that the model $\mathcal{X}$ is not the minimal regular model of X. The right-most component is a -1 component and can be contracted.

Example A.2 (Example 1.10, revisited)**.** Let X be the hyperelliptic curve with affine equation $y^2 = f(x)$, where $f(x)$ is the minimal polynomial of $\alpha := \pi_K^{1/3} + \pi_K^{1/2}$ (for some choice of cube root and square root). By Proposition 4.9(iii), f is a proper key polynomial over a valuation $v_f = [v_0, \ldots, v_n(\varphi_n) = \lambda_n]$, where $D(\varphi_n, \lambda_n)$ is the smallest diskoid containing α and the closest element $\beta \in \overline{K}$ of lower degree over K than 6. We can take $\beta = \pi_K^{1/3}$ and $\varphi_n = x^3 - \pi_K$

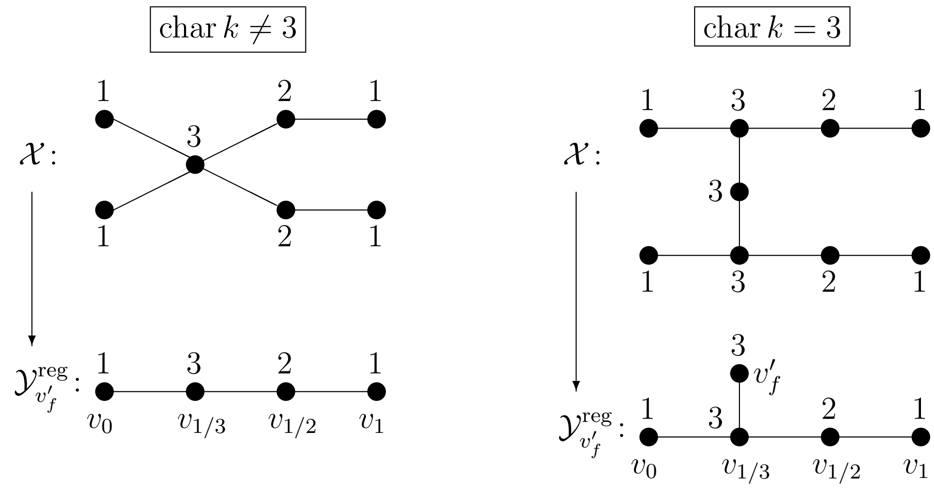


FIGURE 5. The dual graph of the models $\mathcal{X}$ and $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ in Example A.2. Components are labeled with their multiplicities. Components of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ are also labeled with their corresponding valuations.

its minimal polynomial. Since φ_n is a proper key polynomial over $[v_0,\, v_1(x) = 1/3]$, we have $n = 2$ and $v_f = [v_0,\, v_1(x) = 1/3,\, v_2(\varphi_2) = \lambda_2]$, where $\varphi_2 = x^3 - \pi_K$ and $\lambda_2 = \nu_K(\varphi_2(\alpha))$. In particular,

$$\lambda_2 = \begin{cases} 7/6 & \mathrm{char}\, k \neq 3 \\ 3/2 & \mathrm{char}\, k = 3. \end{cases}$$

The successor valuation v'_f to v_f is $[v_0,\, v_1(x) = 1/3]$ when $\mathrm{char}\, k \neq 3$, and is $[v_0,\, v_1(x) = 1/3,\, v_2(\varphi_2) = 4/3]$ when $\mathrm{char}\, k = 3$.

Assume $\mathrm{char}\, k \neq 3$. Applying Proposition 5.12, the minimal regular model $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ including v'_f includes exactly the valuations $v_\lambda = [v_0, v_1(x) = \lambda]$ for $\lambda \in \{0, 1/3, 1/2, 1\}$, just as in Example A.1. The horizontal part of $\mathrm{div}_0(f)$ is regular on this model, and the corresponding irreudicible components of the special fiber have multiplicities 0, 3, 2, and 1 respectively.

The roots of f all have valuation 1/3, so $f(x) = x^6 + a_5x^5 + \cdots + a_0$, with $\nu_K(a_0) = 2$ and $\nu_K(a_i) \geq 2 - i/3$. Thus

$$v_0(f) = 0, v_{1/3}(f) = 2, \text{ and } v_{1/2}(f) = v_1(f) = 2.$$

So $\mathrm{div}_0(f)$ has no odd vertical components, and the normalization $\mathcal{X}$ of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ in $K(X)$ is regular.

Now assume $\mathrm{char}\, k = 3$. Now $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ includes not only the valuations from the $\mathrm{char}\, k \neq 3$ case, but also the valuation $v'_f = [v_0,\, v_1(x) = 1/3,\, v_2(\varphi_2) = 4/3]$. Since f is a key polynomial over v_f, Lemma 4.3(ii) implies that f has φ_2-adic expansion $\varphi_2^2 + b_1\varphi_1 + b_0$, with $v_1(b_1\varphi_1) \geq 3$ and $v_1(b_0) = 3$. Thus $v_1(b_1) \geq 3/2$ and $v'_f(f) = 8/3$. So $\mathrm{div}_0(f)$ still has no odd vertical components, and the normalization $\mathcal{X}$ of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ in $K(X)$ is regular. The dual graphs are shown in Figure 5 (compare to Figure 2 in Example 1.10). In both cases, $\mathcal{X}$ has no -1-components, so it is the minimal regular model.

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Baruch College
Current address: 1 Bernard Baruch Way. New York, NY 10010, USA
Email address: andrewobus@gmail.com

University of Georgia
Current address: 452 Boyd Graduate Studies, 1023 D. W. Brooks Drive, Athens, GA 30602.
Email address: Padmavathi.Srinivasan@uga.edu