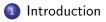
Towards a unified theory of canonical heights on abelian varieties

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Rational points and Galois representations May 10, 2021

Outline



2 Canonical height machine on abelian varieties

- 3 *p*-adic adelic metrics and a canonical *p*-adic height
- 4 Alternate explanation for Q.C. for rational points

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Question: What makes this work? Answer 1 [Balakrishnan-Dogra]: Uses *p*-adic Hodge theory.

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Question: What makes this work? Answer 1 [Balakrishnan-Dogra]: Uses *p*-adic Hodge theory.

Answer 2 [Besser-Mueller-S]: Give an Arakelov-theoretic explanation by a new theory of canonical *p*-adic height functions.

Brief history of Quadratic Chabauty

Why assume r = g? If r < g, then the Chabauty-Coleman method applies. We can take $f = \int \omega$ such that f vanishes on $\overline{J(\mathbb{Q})}$.

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Landmarks in applying Quadratic Chabauty

- Integral points on monic odd-degree hyperelliptic curves. Balakrishnan-Besser-Mueller /Q, 2016. Balakrishnan-Besser-Bianchi-Mueller /number fields, 2020.
- Rational points for curves when rank(NS(J)) > 1.
 Balakrishnan-Dogra, 2016.
- Rational points on the cursed curve X_s(13).
 Balakrishnan-Dogra-Mueller-Tuitman-Vonk, 2019.

Quadratic Chabauty wishlist

 X/\mathbb{Q} nice curve. $b \in X(\mathbb{Q})$. rank $(J(\mathbb{Q})) = r = g$. p good prime.

Want:

$$h=\sum h_{q}\colon J(\mathbb{Q}) o \mathbb{Q}_{p}$$
 such that

- *h* is a quadratic function. *r* = *g* ⇒ *h* can be expanded in an explicit basis of products of single Coleman integrals.
- h_p is an (iterated) Coleman integral.
- For $q \neq p$, h_q takes on finitely many values T on $X(\mathbb{Q}_q)$. Furthermore, $h_q = 0$ if X has potential good reduction at q.
- $h h_p$ locally non-constant function.

A new theory of canonical heights on abelian varieties

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Answer: Yes!

Theorem (Besser-Mueller-S.) Let $\mathcal{L} \in \text{Pic}(J)$. There is a definition of a canonical p-adic height $h_{\mathcal{L}}^{\text{can}} \colon J(\mathbb{Q}) \to \mathbb{Q}_{p}.$

Assume that $[\mathcal{L}] \neq 0 \in \mathsf{NS}(J)$ and that $i^*\mathcal{L} \cong \mathcal{O}_X$. Then $h_{\mathcal{L}}^{\operatorname{can}}$ satisfies the conditions in the Quadratic Chabauty wishlist.

Canonical *p*-adic heights for line bundles on abelian varieties

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Strategy:

- Define a notion of a *p*-adic adelic metric associated to a line bundle *L* on *J* equipped with a "curvature form".
- Identify a canonical metric for a given curvature form.
- Show Quadratic Chabauty wishlist is satisfied using properties of the canonical metric for *L* with [*L*] ≠ 0 and *i*^{*}(*L*) ≅ *O*_X.

History of various constructions of *p*-adic height pairings NEW! One curvature form to rule them all!

Zarhin, 1987. Schneider, 1982. Mazur-Tate, 1983. Coleman-Gross, 1989. Nekovar, 1993.

Question: Why a new theory of canonical *p*-adic heights?

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Question: Why a new theory of canonical *p*-adic heights?

Answer:

- Our construction parallels Zhang's construction of canonical **R**-valued heights from **R**-valued adelic metrics.
- It connects *p*-adic heights for various *p*.
- New way to construct and compute local contributions at finite places to canonical R-valued heights?

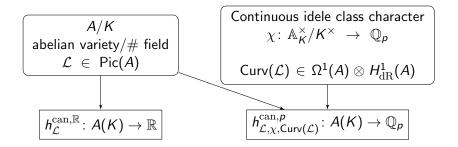
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Canonical height machines on abelian varieties



If \mathcal{L} is symmetric, i.e., if $\mathcal{L} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{can}$ is quadratic:

$$h_{\mathcal{L}}^{\operatorname{can}}(nP) = n^2 h_{\mathcal{L}}^{\operatorname{can}}(P).$$

If \mathcal{L} is anti-symmetric, i.e., if $\mathcal{L}^{-1} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{\operatorname{can}}$ is linear:

$$h_{\mathcal{L}}^{\operatorname{can}}(nP) = n^{1}h_{\mathcal{L}}^{\operatorname{can}}(P).$$

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Ingredients for defining *p*-adic heights

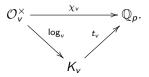
Idele class character of a number field K

A continuous idele class character/global "log" function

$$\chi = \sum \chi_{\mathbf{v}} \colon \mathbb{A}_{\mathbf{K}}^{\times} / \mathbf{K}^{\times} \to \mathbb{Q}_{\mathbf{p}}$$

that is ramified at all primes $\nu \mid p$. This means

- we have $\chi_v(\mathcal{O}_v^{\times}) = 0$ for $v \nmid p$;
- for every $v \mid p$, there is a \mathbb{Q}_p -linear trace map t_v such that we can decompose



Note:

 χ_{v} ramified $\forall v \mid p \Rightarrow$ This factorization extends to K_{v}^{\times} . \rightsquigarrow Get a branch of the logarithm $\log_{v} \colon K_{v}^{\times} \to K_{v}$.

Ingredients for defining *p*-adic height functions Curvature forms for line bundles

② For every v | p, a class $Curv(\mathcal{L}_v) \in \Omega^1(\mathcal{A}_v) \otimes H^1_{dR}(\mathcal{A}_v)$ called the curvature form for the line bundle such that

$$\Omega^{1}(A_{\nu}) \otimes H^{1}_{\mathrm{dR}}(A_{\nu}) \xrightarrow{\cup} H^{2}_{\mathrm{dR}}(A_{\nu})$$
$$\operatorname{Curv}(\mathcal{L}_{\nu}) \mapsto c_{1}(\mathcal{L}_{\nu}).$$

Example: Let X/K be a nice curve of genus $g \ge 1$. Fix a complementary subspace W to $\Omega^1(X_v)$ in $H^1_{dR}(X_v)$. Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $\Omega^1(X_v)$. If $\{\overline{\omega_1}, \ldots, \overline{\omega_g}\}$ be the unique dual basis in W (with respect to the cup product pairing). Then

$$2\sum_{i=1}^{g}\omega_i\otimes\overline{\omega_i}$$

is a curvature form for the tangent bundle of X_{ν} .

From curvature forms to metrics

Proposition: [Besser, *p*-adic Arakelov theory, 2005] For every curvature form $\operatorname{Curv}(\mathcal{L}_{\nu}) \in \Omega^{1}(\mathcal{A}_{\nu}) \otimes \mathcal{H}^{1}_{\operatorname{dR}}(\mathcal{A}_{\nu})$, there is an associated metric $\log_{\mathcal{L}} \in \mathcal{O}_{\operatorname{Col}}(\mathcal{L}_{\nu}^{\times})$ obtained by an iterated integral.

$$\operatorname{Curv}(\mathcal{L}_{\boldsymbol{v}}) := \sum \omega_i \otimes [\eta_i] \mapsto \int \omega_i \left(\int \eta_i \right) =: \log_{\mathcal{L}}(\boldsymbol{s}).$$

We have, as before,

 $\log_{\mathcal{L}}(\alpha w) = \log_{v}(\alpha) + \log_{\mathcal{L}}(w) \quad \text{ for every } \alpha \in \overline{K_{v}}^{\times}, w \in \mathcal{L}_{x}^{\times}.$

Note: There are multiple metrics with the same curvature, but any two such metrics differ by the integral of a holomorphic form.

\mathbb{Q} -valued metrics away from p

Let $v \nmid p$ be a finite place of K. Let $\nu_v = \log || \cdot ||_v$. Let X/K_v be a projective variety.

Definition: (Inspired by Moret-Bailly, Zhang) A (\mathbb{Q} -valued) metric on a line-bundle \mathcal{L} is a locally constant function (for the analytic topology)

$$u \colon \operatorname{\mathsf{Tot}}(\mathcal{L}) \setminus \{0\} =: \mathcal{L}^{ imes} o \mathbb{Q}$$

such that

$$\nu(\alpha w) = \nu_{v}(\alpha) + \nu(w) \qquad \forall \alpha \in \overline{K_{v}}^{\times}, \forall w \in \mathcal{L}_{x}^{\times}, x \in X(\overline{K_{v}})$$

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Note: Can define pull-backs and tensor powers of metrics.

Examples of metrics

Example 1: Model metrics

Let \mathscr{X}/\mathbb{Z}_p be a proper flat integral model of X/\mathbb{Q}_p .

Let \mathscr{L} be a line bundle on \mathscr{X} extending a line bundle \mathcal{L} on X.

Let s be a meromorphic section of \mathcal{L} .

Let $x \in X(\overline{\mathbb{Q}})$ with closure \overline{x} in \mathscr{X} .

The (Q-valued) model metric coming from \mathscr{L} is given by

 $\nu_{\mathscr{L}}(s(x)) :=$ valuation of $x^*(s)$ in the lattice $\overline{x}^*(\mathscr{L})$.

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Example 2: Admissible metrics

Come from a model + "evaluation+norm" of invertible functions. Closed under pull-backs/tensor products.

Admissible metrics on \mathcal{O}_X "factor through the reduction graph".

p-adic adelic-metrics and \mathbb{Q}_p -valued heights

Definition: An *p*-adic adelic metric on a line bundle \mathcal{L} on a projective variety X/K is a collection of metrics

 $\{\nu_{v} \text{ on } \mathcal{L}_{v}/X_{v}/K_{v} \colon v \nmid p \text{ a place of } K\} \cup \{\log_{\mathcal{L}_{v}} \colon v \mid p\}.$

such that ν_v is \mathbb{Q} -valued for every $v \nmid p$ and in addition a model-metric for almost every place v.

Definition: The *p*-adic height function *h* associated to a *p*-adic adelic metric on a line bundle \mathcal{L} on *X* as above is

$$h: X(K) \to \mathbb{Q}_p$$
$$x \mapsto \sum_{v \nmid p} \nu_v(s(x))\chi_v(\pi_v) + \sum_{v \mid p} \log_{\mathcal{L}}(s(x)),$$

for some choice of section $s \in \mathcal{L}_x^{\times}$.

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Quadratic Chabauty wishlist + *p*-adic adelic metrics

Want:

- A *p*-adic adelic metric such that the associated height function h is a quadratic function on J(K).
- So For all $v \nmid p$, we want the pull-back of h_v to X(K) under Abel-Jacobi map *i* to take on finitely many values.
- **③** Want $h h_p$ to be a locally non-constant Coleman function.

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Solution:

Choose a *p*-adic adelic metric *h* on $\mathcal{L} \in Pic(J)$ such that

- **1** *h* is a canonical *p*-adic adelic metric on \mathcal{L} .
- $i^*(\mathcal{L}) \cong \mathcal{O}_X \Rightarrow i^* h_v \text{ is an admissible metric on } \mathcal{O}_X.$
- **3** $[\mathcal{L}]$ is nonzero in NS(*J*).

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Step 3: For $v \nmid p$, use the canonical \mathbb{Q} -valued metric from the canonical \mathbb{R} -valued height.

Note: Can relax the assumption that p is a good prime by replacing Coleman integrals by Vologodsky integrals.

 $\Rightarrow \log_{\mathcal{L},p} = \int (\omega \int \eta)$ makes sense even when p is a bad prime.

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Let $h^{\ell} = (h_{v}^{\ell})_{v}$ be the canonical \mathbb{Q}_{ℓ} -valued height. Question: Is there any connection between h_{p}^{ℓ} and $\log_{\mathcal{L},p} =: h_{p}^{p}$?

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Coefficient of $\log(p)$ in $\log_{\mathcal{L},p}$ at a point equals the value of h_p^{ℓ} . \Rightarrow Curvature controls/unifies the local contributions at all places!