# Towards a unified theory of canonical heights on abelian varieties

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Rational points and Galois representations
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# Outline

# $p$-adic functions vanishing on rational points

Let $X$ be a nice curve over $\mathbb{Q}$ of genus $g$ with good reduction at $p$.
Let $J$ be the Jacobian of $X$ with Mordell-Weil rank $r$.

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$f\colon X(\mathbb{Q}_p) \to \mathbb{Q}_p$ such that $X(\mathbb{Q}) \subset \{x\colon f(x) \in T, \; T \text{ finite}\}$.

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Answer 1 [Balakrishnan-Dogra]: Uses $p$-adic Hodge theory.

Answer 2 [Besser-Mueller-S]: Give an Arakelov-theoretic explanation by a new theory of canonical $p$-adic height functions.

Why assume $r = g$?
If $r < g$, then the Chabauty-Coleman method applies.
We can take $f = \int \omega$ such that $f$ vanishes on $\overline{J(\mathbb{Q})}$.

# Brief history of Quadratic Chabauty

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## Landmarks in applying Quadratic Chabauty

1. Integral points on monic odd-degree hyperelliptic curves.
   Balakrishnan-Besser-Mueller $/\mathbb{Q}$, 2016.
   Balakrishnan-Besser-Bianchi-Mueller /number fields, 2020.
2. Rational points for curves when $\mathrm{rank}(\mathrm{NS}(J)) > 1$.
   Balakrishnan-Dogra, 2016.
3. Rational points on the cursed curve $X_s(13)$.
   Balakrishnan-Dogra-Mueller-Tuitman-Vonk, 2019.

# Quadratic Chabauty wishlist

$X/\mathbb{Q}$ nice curve. $b \in X(\mathbb{Q})$. $\mathrm{rank}(J(\mathbb{Q})) = r = g$. $p$ good prime.

Want:

$$h = \sum h_q \colon J(\mathbb{Q}) \to \mathbb{Q}_p \quad \text{such that}$$

- $h$ is a quadratic function. $r = g \Rightarrow h$ can be expanded in an explicit basis of products of single Coleman integrals.
- $h_p$ is an (iterated) Coleman integral.
- For $q \neq p$, $h_q$ takes on finitely many values $T$ on $X(\mathbb{Q}_q)$. Furthermore, $h_q = 0$ if $X$ has potential good reduction at $q$.
- $h - h_p$ locally non-constant function.

# A new theory of canonical heights on abelian varieties

$X/\mathbb{Q}$ nice curve.
$b \in X(\mathbb{Q})$. $i\colon X \to J$ Abel-Jacobi map.
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Answer: Yes!

**Theorem (Besser-Mueller-S.)**

*Let $\mathcal{L} \in \mathrm{Pic}(J)$. There is a definition of a canonical $p$-adic height*

$$h_{\mathcal{L}}^{\mathrm{can}}\colon J(\mathbb{Q}) \to \mathbb{Q}_p.$$

*Assume that $[\mathcal{L}] \neq 0 \in \mathrm{NS}(J)$ and that $i^*\mathcal{L} \cong \mathcal{O}_X$. Then $h_{\mathcal{L}}^{\mathrm{can}}$ satisfies the conditions in the Quadratic Chabauty wishlist.*

# Canonical $p$-adic heights for line bundles on abelian varieties

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Strategy:

- Define a notion of a $p$-adic adelic metric associated to a line bundle $\mathcal{L}$ on $J$ equipped with a "curvature form".
- Identify a canonical metric for a given curvature form.
- Show Quadratic Chabauty wishlist is satisfied using properties of the canonical metric for $\mathcal{L}$ with $[\mathcal{L}] \neq 0$ and $i^*(\mathcal{L}) \cong \mathcal{O}_X$.

# History of various constructions of $p$-adic height pairings

## NEW! One curvature form to rule them all!

Zarhin, 1987.
Schneider, 1982.
Mazur-Tate, 1983.
Coleman-Gross, 1989.
Nekovar, 1993.

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Question: Why a new theory of canonical $p$-adic heights?

Answer:

- Our construction parallels Zhang's construction of canonical $\mathbb{R}$-valued heights from $\mathbb{R}$-valued adelic metrics.
- It connects $p$-adic heights for various $p$.
- New way to construct and compute local contributions at finite places to canonical $\mathbb{R}$-valued heights?

$A/K$
abelian variety/# field
$\mathcal{L} \in \mathrm{Pic}(A)$

Continuous idele class character
$\chi\colon \mathbb{A}_K^\times/K^\times \to \mathbb{Q}_p$

$\mathrm{Curv}(\mathcal{L}) \in \Omega^1(A) \otimes H^1_{\mathrm{dR}}(A)$

$h_{\mathcal{L}}^{\mathrm{can},\mathbb{R}}\colon A(K) \to \mathbb{R}$

$h_{\mathcal{L},\chi,\mathrm{Curv}(\mathcal{L})}^{\mathrm{can},p}\colon A(K) \to \mathbb{Q}_p$

If $\mathcal{L}$ is symmetric, i.e., if $\mathcal{L} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{\mathrm{can}}$ is quadratic:

$$h_{\mathcal{L}}^{\mathrm{can}}(nP) = n^2 h_{\mathcal{L}}^{\mathrm{can}}(P).$$

If $\mathcal{L}$ is anti-symmetric, i.e., if $\mathcal{L}^{-1} \cong [-1]^*(\mathcal{L})$, then $h_{\mathcal{L}}^{\mathrm{can}}$ is linear:

$$h_{\mathcal{L}}^{\mathrm{can}}(nP) = n^1 h_{\mathcal{L}}^{\mathrm{can}}(P).$$

# Outline

# Ingredients for defining $p$-adic heights

## Idele class character of a number field $K$

1. A continuous idele class character/global "log" function

$$\chi = \sum \chi_v \colon \mathbb{A}_K^\times / K^\times \to \mathbb{Q}_p$$

that is ramified at all primes $\nu \mid p$. This means

- we have $\chi_v(\mathcal{O}_v^\times) = 0$ for $v \nmid p$;
- for every $v \mid p$, there is a $\mathbb{Q}_p$-linear trace map $t_v$ such that we can decompose

$$\begin{array}{ccc} \mathcal{O}_v^\times & \xrightarrow{\quad \chi_v \quad} & \mathbb{Q}_p. \\ & {\scriptstyle \log_v} \searrow \quad \nearrow {\scriptstyle t_v} & \\ & K_v & \end{array}$$

Note:

$\chi_v$ ramified $\forall v \mid p \Rightarrow$ This factorization extends to $K_v^\times$.

$\rightsquigarrow$ Get a branch of the logarithm $\log_v \colon K_v^\times \to K_v$.

2. For every $v \mid p$, a class $\mathrm{Curv}(\mathcal{L}_v) \in \Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v)$ called the curvature form for the line bundle such that

$$\Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v) \xrightarrow{\cup} H^2_{\mathrm{dR}}(A_v)$$
$$\mathrm{Curv}(\mathcal{L}_v) \mapsto c_1(\mathcal{L}_v).$$

Example: Let $X/K$ be a nice curve of genus $g \geq 1$.
Fix a complementary subspace $W$ to $\Omega^1(X_v)$ in $H^1_{\mathrm{dR}}(X_v)$.
Let $\{\omega_1, \ldots, \omega_g\}$ be a basis for $\Omega^1(X_v)$.
If $\{\overline{\omega_1}, \ldots, \overline{\omega_g}\}$ be the unique dual basis in $W$ (with respect to the cup product pairing). Then

$$2 \sum_{i=1}^{g} \omega_i \otimes \overline{\omega_i}$$

is a curvature form for the tangent bundle of $X_v$.

# From curvature forms to metrics

Proposition: [Besser, $p$-adic Arakelov theory, 2005]
For every curvature form $\mathrm{Curv}(\mathcal{L}_v) \in \Omega^1(A_v) \otimes H^1_{\mathrm{dR}}(A_v)$, there is an associated metric $\log_{\mathcal{L}} \in \mathcal{O}_{\mathrm{Col}}(\mathcal{L}_v^\times)$ obtained by an iterated integral.

$$\mathrm{Curv}(\mathcal{L}_v) := \sum \omega_i \otimes [\eta_i] \mapsto \int \omega_i \left( \int \eta_i \right) =: \log_{\mathcal{L}}(s).$$

We have, as before,

$$\log_{\mathcal{L}}(\alpha w) = \log_v(\alpha) + \log_{\mathcal{L}}(w) \quad \text{for every } \alpha \in \overline{K_v}^\times, w \in \mathcal{L}_x^\times.$$

Note: There are multiple metrics with the same curvature, but any two such metrics differ by the integral of a holomorphic form.

## $\mathbb{Q}$-valued metrics away from $p$

Let $v \nmid p$ be a finite place of $K$.
Let $\nu_v = \log ||\cdot||_v$.
Let $X/K_v$ be a projective variety.

Definition: (Inspired by Moret-Bailly, Zhang)
A ($\mathbb{Q}$-valued) metric on a line-bundle $\mathcal{L}$ is a locally constant function (for the analytic topology)

$$\nu\colon \operatorname{Tot}(\mathcal{L}) \setminus \{0\} =: \mathcal{L}^\times \to \mathbb{Q}$$

such that

$$\nu(\alpha w) = \nu_v(\alpha) + \nu(w) \qquad \forall \alpha \in \overline{K_v}^\times, \forall w \in \mathcal{L}_x^\times, x \in X(\overline{K_v})$$

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Note: Can define pull-backs and tensor powers of metrics.

Example 1: Model metrics

Let $\mathscr{X}/\mathbb{Z}_p$ be a proper flat integral model of $X/\mathbb{Q}_p$.

Let $\mathscr{L}$ be a line bundle on $\mathscr{X}$ extending a line bundle $\mathcal{L}$ on $X$.

Let $s$ be a meromorphic section of $\mathcal{L}$.

Let $x \in X(\overline{\mathbb{Q}})$ with closure $\overline{x}$ in $\mathscr{X}$.

The ($\mathbb{Q}$-valued) model metric coming from $\mathscr{L}$ is given by

$$\nu_{\mathscr{L}}(s(x)) := \text{ valuation of } x^*(s) \text{ in the lattice } \overline{x}^*(\mathscr{L}).$$

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Example 2: Admissible metrics

Come from a model + "evaluation+norm" of invertible functions.
Closed under pull-backs/tensor products.
Admissible metrics on $\mathcal{O}_X$ "factor through the reduction graph".

# $p$-adic adelic-metrics and $\mathbb{Q}_p$-valued heights

Definition: An $p$-adic adelic metric on a line bundle $\mathcal{L}$ on a projective variety $X/K$ is a collection of metrics

$$\{\nu_v \text{ on } \mathcal{L}_v/X_v/K_v \colon \nu \nmid p \text{ a place of } K\} \cup \{\log_{\mathcal{L}_v} \colon v \mid p\}.$$

such that $\nu_v$ is $\mathbb{Q}$-valued for every $v \nmid p$ and in addition a model-metric for almost every place $v$.

Definition: The $p$-adic height function $h$ associated to a $p$-adic adelic metric on a line bundle $\mathcal{L}$ on $X$ as above is

$$\begin{aligned} h \colon X(K) &\to \mathbb{Q}_p \\ x &\mapsto \sum_{v \nmid p} \nu_v(s(x))\chi_v(\pi_v) + \sum_{v \mid p} \log_{\mathcal{L}}(s(x)), \end{aligned}$$

for some choice of section $s \in \mathcal{L}_x^\times$.

# Outline

# Quadratic Chabauty wishlist + $p$-adic adelic metrics

Want:

1. A $p$-adic adelic metric such that the associated height function $h$ is a quadratic function on $J(K)$.
2. For all $v \nmid p$, we want the pull-back of $h_v$ to $X(K)$ under Abel-Jacobi map $i$ to take on finitely many values.
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Solution:

Choose a $p$-adic adelic metric $h$ on $\mathcal{L} \in \mathrm{Pic}(J)$ such that

1. $h$ is a canonical $p$-adic adelic metric on $\mathcal{L}$.
2. $i^*(\mathcal{L}) \cong \mathcal{O}_X \Rightarrow i^* h_v$ is an admissible metric on $\mathcal{O}_X$.
3. $[\mathcal{L}]$ is nonzero in $\mathrm{NS}(J)$.

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# Canonical metrics on line bundles on abelian varieties

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Step 3: For $v \nmid p$, use the canonical $\mathbb{Q}$-valued metric from the canonical $\mathbb{R}$-valued height.

Note: Can relax the assumption that $p$ is a good prime by replacing Coleman integrals by Vologodsky integrals.

$\Rightarrow \log_{\mathcal{L},p} = \int \left(\omega \int \eta\right)$ makes sense even when $p$ is a bad prime.

# Local heights away from $p$ using $p$-adic heights at $p$

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Question: Is there any connection between $h_p^\ell$ and $\log_{\mathcal{L},p} =: h_p^p$?

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Coefficient of $\log(p)$ in $\log_{\mathcal{L},p}$ at a point equals the value of $h^\ell_p$.
$\Rightarrow$ Curvature controls/unifies the local contributions at all places!