Conductors and minimal discriminants of hyperelliptic curves: a comparison in the tame case

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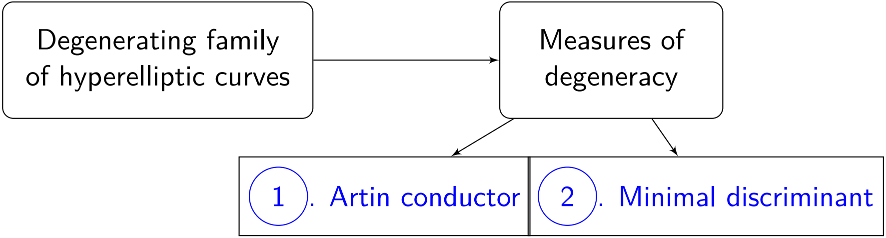
What are conductors and minimal discriminants?



Main Question: How are measures (1) and (2) related? Inequality?

Example: $X/\mathbb{C}((t))$, $y^2 = f(x)$, genus $g = 3$ hyperelliptic

$$f(x) = (x^2 - t)(x^2 - 2t)(x-1)(x-1+t)(x-1+2t)(x-1+3t)$$

$$-\operatorname{Art}(X) = 8, \qquad \Delta_X = 18.$$

How are conductors and minimal discriminants related?

Earlier results: (small genus, all residue characteristics)

- If $g = 1$, then $-\mathrm{Art}(X) = \Delta_X$. [Ogg-Saito formula]
- If $g = 2$, then Liu showed that $-\mathrm{Art}(X) \leq \Delta_X$. He showed that equality does not always hold.

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Question: Does $-\mathrm{Art}(X) \leq \Delta_X$ hold for hyperelliptic curves of arbitrary genus g?
Today: Yes, if the residue characteristic is $> 2g + 1$. [S.]

Notation

R : $\mathbb{C}[[t]]$

K : $\mathbb{C}((t))$, field of Laurent series

$\overline{K}$: $\cup_{n\geq 1} \mathbb{C}((t^{1/n}))$, field of Puiseux series

$\mathrm{ord}_{t=0}$: t-adic valuation $\overline{K} \to \mathbb{Q} \cup \{\infty\}$,
normalized using $\mathrm{ord}_{t=0}(t) = 1$.

X : smooth hyperelliptic K-curve

g : genus of X

Definition: The minimal discriminant Δ_X of X/K is the nonnegative integer

$$\Delta_X := \min_{\substack{f(x)\in R[x] \\ y^2=f(x),\ \text{eqn. for X}}} \operatorname{ord}_{t=0}(\underbrace{\operatorname{disc}(f)}_{\in R}).$$

An example:

$$C_1 : y^2 = x(x-t)(x-2t)(x-3t) \quad \rightsquigarrow \quad \operatorname{disc}(f) = 12.$$
$$C_2 : y'^2 = x'(x'-1)(x'-2)(x'-3) \quad \rightsquigarrow \quad \operatorname{disc}(f) = 0.$$

Here $C_1 \cong_K C_2$ via $x' = \frac{x}{t}, y' = \frac{y}{t^2} \rightsquigarrow \Delta_X = 0$.

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$$-\operatorname{Art}(\mathcal{X}) := \chi^{\text{top}}(\underbrace{\mathcal{X}_{t=0}}_{\text{curve over } \mathbb{C}}) - (2 - 2g).$$

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Definition: The (negative of the) Artin conductor $-\operatorname{Art}(X)$ of X/K is the nonnegative integer

$$-\operatorname{Art}(X) := \min_{\substack{\mathcal{X}\to\operatorname{Spec} R \\ \text{proper, regular} \\ \text{model for } X/K}} [-\operatorname{Art}(\mathcal{X})] \;\; (= -\operatorname{Art}(\mathcal{X}^{\min})).$$

Remark: If P is a closed point of $\mathcal{X}$, and $\operatorname{Bl}_P(\mathcal{X})$ is the blowup of $\mathcal{X}$ at P, then $-\operatorname{Art}(\operatorname{Bl}_P(\mathcal{X})) = [-\operatorname{Art}(\mathcal{X})] + 1$.

Why care about conductors and minimal discriminants?

Fact: The invariants

$$\Delta_X = [-\operatorname{Art}(X)] = 0$$

if and only if X has a smooth model over $\operatorname{Spec} R$ and are strictly > 0 otherwise.

Theorem (S.)

Let K be the fraction field of a Henselian discrete valuation ring. Let X be a smooth hyperelliptic curve over K of genus $g \geq 1$. Assume that the residue characteristic is $> 2g+1$. Then,

$$-\operatorname{Art}(X) \leq \Delta_X.$$

Computing conductors

$$
\begin{array}{ccc}
\mathrm{Norm}(\mathcal{Y}, K(X)) =: \mathcal{X} & \longleftarrow & X \\
\big\downarrow \text{finite} & & \big\downarrow 2{:}1 \\
B \subset \mathcal{Y} & \longleftarrow & \mathbb{P}^1_K \\
\text{Branch divisor} \quad \big\downarrow & & \big\downarrow \\
\mathrm{Spec}\, R & \longleftarrow & \mathrm{Spec}\, K
\end{array}
$$

Riemann-Hurwitz formula:

$$-\,\mathrm{Art}(\mathcal{X}) = 2\chi^{\mathrm{top}}(\mathcal{Y}_{t=0}) - \chi^{\mathrm{top}}(B_{t=0}) - (2-2g).$$

Example: $\mathcal{Y} = \mathbb{P}^1_R,\ f(x) = x^{2g+2} - t,\ B\colon f = 0.$

$$\chi^{\mathrm{top}}(\mathcal{Y}_{t=0}) =$$

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$$-\,\mathrm{Art}(\mathcal{X}) =$$

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$$\chi^{\mathrm{top}}(B_{t=0}) = \chi^{\mathrm{top}}(\mathrm{pt}) = 1.$$
$$-\operatorname{Art}(\mathcal{X}) = 2 \cdot 2 - 1 - (2-2g) = 2g+1.$$

Goal: Visualize relative distances of roots $\rightsquigarrow$
Reorganize roots of $f \in R[x]$ into a metric tree $T(f)$

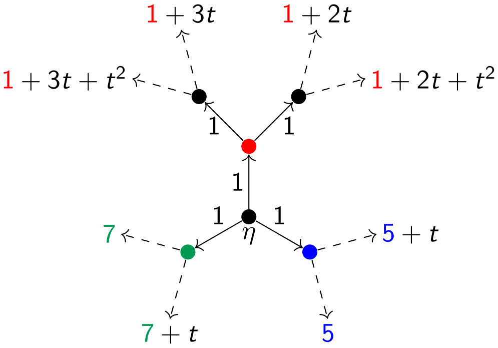
Example 1: Roots of f are K-rational

$$f(x) = (x-7)(x-7-t)(x-5)(x-5-t)$$
$$(x-1-2t)(x-1-2t-t^2)(x-1-3t)(x-1-3t-t^2)$$

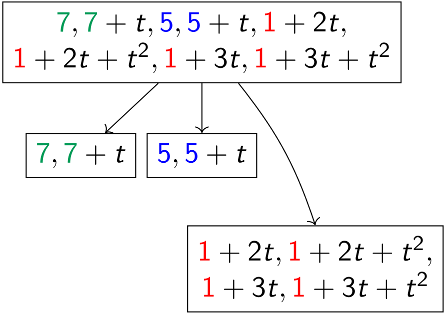
Roots of $f = \{7, 7+t, 5, 5+t, 1+2t, 1+2t+t^2, 1+3t, 1+3t+t^2\}$

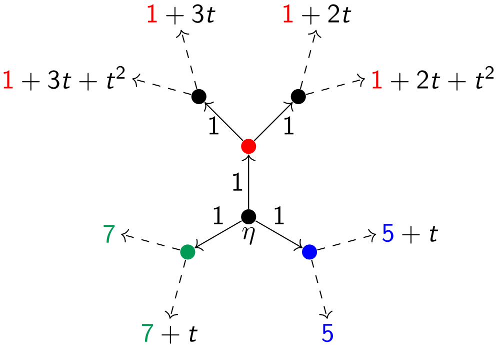
Idea: Partition the roots successively, using their residues mod t, mod t^2, etc..

$$7, 7+t, 5, 5+t, 1+2t,$$
$$1+2t+t^2, 1+3t, 1+3t+t^2$$



Metric tree of a polynomial





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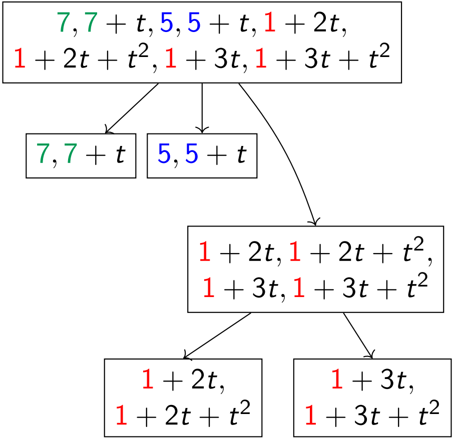
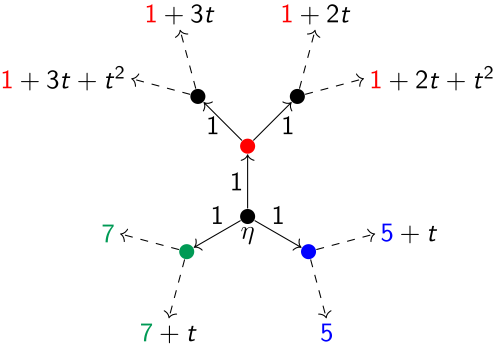




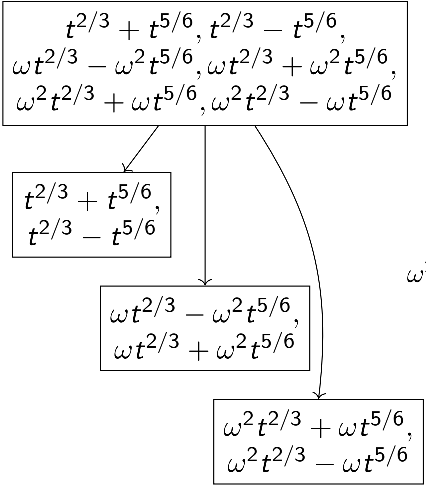
Example 2: Roots of f are not K-rational

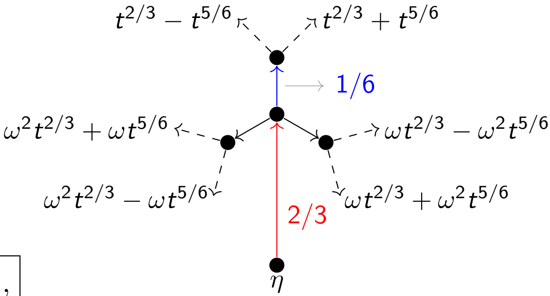
Roots of $f = t^{2/3} + t^{5/6}$ and all its Galois conjugates

$$= \{t^{2/3} + t^{5/6}, t^{2/3} - t^{5/6}, \omega t^{2/3} - \omega^2 t^{5/6}, \omega t^{2/3} + \omega^2 t^{5/6}, \omega^2 t^{2/3} + \omega t^{5/6}, \omega^2 t^{2/3} - \omega t^{5/6}\}$$

- The roots of f have $t^{1/6}$-adic expansions.
- Draw the metric tree using $t^{1/6}$-adic expansions.
- Rescale all lengths by $1/6$.

Metric tree of a polynomial

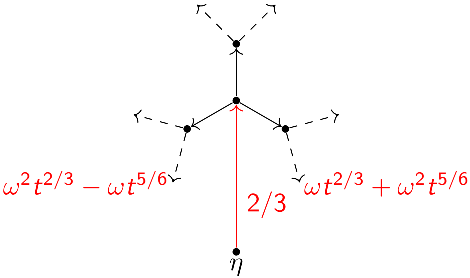


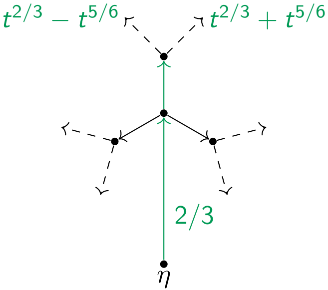


The metric function and discriminants

Definition Let α, β be two roots of f.

$$(\alpha|\beta)_\eta := \mathrm{ord}_{t=0}\ (\alpha - \beta)$$
$$= \text{Length of common segment of path from } \eta \text{ to } \alpha \text{ and } \beta.$$





$(\omega^2 t^{2/3} - \omega t^{5/6} | \omega t^{2/3} + \omega^2 t^{5/6})_\eta = 2/3$

$(t^{2/3} - t^{5/6} | t^{2/3} + t^{5/6})_\eta = 5/6$

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Fact Let (α_i) be the collection of roots of f. Then,

$$\Delta_f = \sum_{i \neq j} (\alpha_i|\alpha_j)_\eta.$$

Remark: Suffices to find ONE proper regular model $\mathcal{X}$ such that

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Our choice: Jung's method for resolving surface singularties. Let

$$\begin{aligned} \Delta_f &:= \operatorname{ord}_{t=0}(\operatorname{disc}(f)) \text{ for any } f \in R[x] \\ \operatorname{div}(f) &:= \text{Divisor of } f \text{ in } \mathbb{P}^1_R \\ \mathcal{Y}_f &:= \text{Embedded resolution of the pair } (\mathbb{P}^1_R, \operatorname{div}(f)) \\ \mathcal{X}_f &:= \text{Normalization of } \mathcal{Y}_f \text{ in } K(X) \end{aligned}$$

Jung $\Rightarrow$ The model $\mathcal{X}_f \to \operatorname{Spec} R$ of X is (almost) regular.

Explicit regular model: Let $y^2 = f(x)$ be an equation for X with $f(x) \in R[x]$ and $\Delta_X = \Delta_f$. Set $\mathcal{X} = \mathcal{X}_f$.

Overview of proof for $-\operatorname{Art}(\mathcal{X}_f) \leq \Delta_f$.

Strategy: Induction on Δ_f.

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STEP 3: Let $(-\operatorname{Art}(\mathcal{X}_{f_P})) = (-\operatorname{Art}(\mathcal{X}_{f_P^{\mathrm{sm}}})) + (-\operatorname{Art}(\mathcal{X}_{f_P^{\mathrm{nod}}}))$, and let $\Delta_{f_P} = \Delta_{f_P^{\mathrm{sm}}} + \Delta_{f_P^{\mathrm{nod}}}$.
3A: Compute $\delta(-\operatorname{Art}) := -\operatorname{Art}(\mathcal{X}_f) - \sum_{P \in \operatorname{div}(f)^{\mathrm{Sing}}} (-\operatorname{Art}(\mathcal{X}_{f_P}))$.
3B: Compute $\delta(\Delta) := \Delta_f - \sum_{P \in \operatorname{div}(f)^{\mathrm{Sing}}} \Delta_{f_P}$.
3C: Prove $\delta(-\operatorname{Art}) \leq \delta(\Delta)$.

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STEP 4: Termination: Reduce to $\operatorname{div}(f)$ regular after finitely many replacements.

Outline

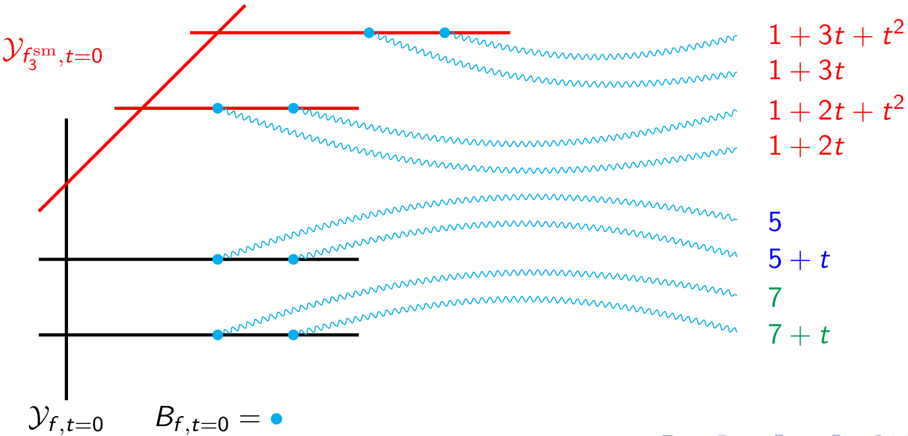
Main ingredients in calculation:

- Jung's method gives a natural way to identify $(\mathcal{Y}_{f_P^{\mathrm{sm/nod}},t=0}, B_{f_P^{\mathrm{sm/nod}},t=0})$ with a closed subset of $(\mathcal{Y}_{t=0}, B_{t=0})$.
- Use the additivity of χ^{top} and inclusion-exclusion/excision for $(\mathcal{Y}_{f_P^{\mathrm{sm/nod}},t=0}, B_{f_P^{\mathrm{sm/nod}},t=0}) \subset (\mathcal{Y}_{t=0}, B_{t=0})$.

$\rightsquigarrow$ an explicit formula for $\delta(-\mathrm{Art}) :=$
$-\mathrm{Art}(\mathcal{X}_f) - \sum_{P\in \mathrm{div}(f)^{\mathrm{Sing}}}[(-\mathrm{Art}(\mathcal{X}_{f_P^{\mathrm{sm}}})) + (-\mathrm{Art}(\mathcal{X}_{f_P^{\mathrm{nod}}}))]$
(Omitted).

$(\mathcal{Y}_{f_P,t=0}, B_{f_P,t=0})$ is a closed subset of $(\mathcal{Y}_{f,t=0}, B_{f,t=0})$

$$f(x) = (x-7)(x-7-t)(x-5)(x-5-t)$$
$$(x-1-2t)(x-1-2t-t^2)(x-1-3t)(x-1-3t-t^2)$$
$$f_3^{\mathrm{sm}}(x) = (x-2)(x-2-t)(x-3)(x-3-t).$$



Outline

Remarks:

- It is enough to relate $T(f)$ to $T(f_P^{\mathrm{sm}})$ and $T(f_P^{\mathrm{nod}})$.
- If we do it carefully, we simultaneously obtain a proof of termination of induction.

$T(f_P^{\mathrm{sm}})$ is a subtree of $T(f)$.

$T(f_P^{\mathrm{nod}})$ is more tricky!

$T(f_P^{\mathrm{sm}})$ is a subtree of $T(f)$

$$f(x) = (x-7)(x-7-t)(x-5)(x-5-t)$$
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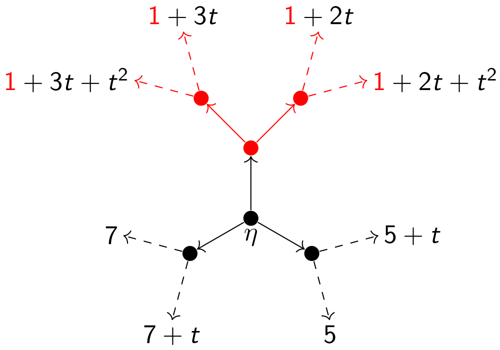


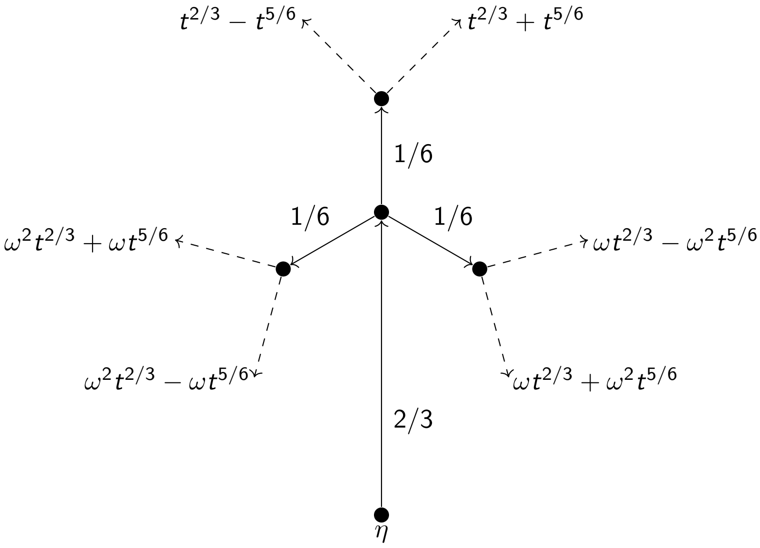
Obtaining $T(f_P^{\mathrm{nod}})$ from Abhyankar's Inversion Formula

Special case: Suppose that

- f is irreducible,
- the valuation of a root of f is $a/b < 1$, and,
- $\gcd(a, b) = 1$.

Fact: $T(f)$ is obtained by gluing b identical subtrees at distance a/b from η.

Example: Roots of $f = t^{2/3} + t^{5/6}$ and its conjugates, $a/b = 2/3$.



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Abhyankar $\Rightarrow$ $T(f_P^{\mathrm{nod}})$ can then be obtained by gluing a identical new subtrees at distance $(b/a) - 1$ from η, where

$$\text{New subtree metric} = (\text{Old subtree metric}) \cdot b/a.$$

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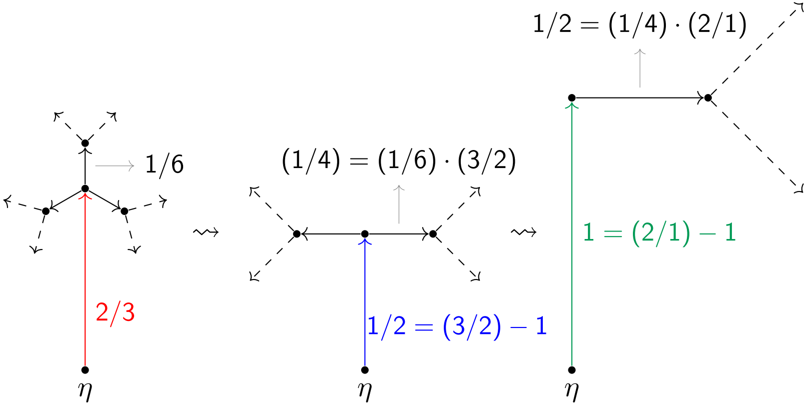
General case: Transform trees for each irreducible nodal factor separately, and then glue them back together maintaining 'expected overlaps'.

Distance from η changes according to $(a/b) \rightsquigarrow (b/a) - 1$.
New subtree metric = (Old subtree metric) $\cdot\ b/a$.



Outline

An explicit example. $f(x) = (x^2 - t)(x^2 - 2t)(x - 1)(x - 1 + t)(x - 1 + 2t)(x - 1 + 3t)$.

Replacement steps:

$$f(x) = (x^2 - t)(x^2 - 2t)(x - 1)(x - 1 + t)(x - 1 + 2t)(x - 1 + 3t)$$

$$f_{P_2}(x) = (x - t)(x - (1/2)t) \qquad f_{P_1}(x) = x(x + 1)(x + 2)(x + 3)$$

$$\downarrow$$

$$f_{P_3}(x) = (x - 1)(x - (1/2))$$

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$$\downarrow$$

$$f_{P_3}(x) = (x-1)(x-(1/2))$$

3A: $\delta(-\mathrm{Art})$: $-\mathrm{Art}(\mathcal{X}_f) - [-\mathrm{Art}(\mathcal{X}_{f_1}) - \mathrm{Art}(\mathcal{X}_{f_2})] = 2 + 4.$
3B: $\delta(\Delta)$: $\Delta_f - [\Delta_{f_1} + \Delta_{f_2}] = 2\binom{4}{2} + 4 = 12 + 4.$
3C: $\delta(-\mathrm{Art}) \leq \delta(\Delta)$: $2 + 4 \leq 12 + 4.$

3A: $\delta(-\mathrm{Art})$: $-\mathrm{Art}(\mathcal{X}_{f_2}) - [-\mathrm{Art}(\mathcal{X}_{f_3})] = 2.$
3B: $\delta(\Delta)$: $\Delta_{f_2} - \Delta_{f_3} = 2\binom{2}{2} = 2.$
3C: $\delta(-\mathrm{Art}) \leq \delta(\Delta)$: $2 \leq 2.$

Thank you!