QUADRATIC ENRICHMENT OF THE LOGARITHMIC DERIVATIVE OF THE ZETA FUNCTION

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Abstract. We define an enrichment of the logarithmic derivative of the zeta function of a variety over a finite field to a power series with coefficients in the Grothendieck–Witt group. This enrichment is related to the topology of the real points of a lift. We show a rationality result for cellular schemes over a field, and compute several examples, including toric varieties.

Contents

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1. Introduction

Let X be a smooth projective variety over a finite field $\mathbb{F}_q$. The zeta function of X is defined by

$$\zeta_X(t) = \exp\left(\sum_{m\geq 1} \frac{|X(\mathbb{F}_{q^m})|}{m} t^m\right).$$

It is known by the Weil conjectures that $\zeta_X(t)$ is a rational function related to the complex points of a lift of X. A proof of its rationality proceeds by using the Grothendieck–Lefschetz trace formula, which replaces each point count $|X(\mathbb{F}_{q^m})|$ by an alternating sum of traces of the m-th power of the Frobenius endomorphism on the ℓ-adic cohomology groups of X (see, for example, [Mil13, Theorem 27.6]). The aim of this paper is to define and study an enrichment of (the logarithmic derivative of) this zeta function, by replacing the point counts with traces in the sense of $\mathbb{A}^1$-homotopy theory.

Let X be a smooth, proper variety over a field k (not necessarily finite). Due to work of Hu, Rioux, and Dubouloz–Déglise–Østvaer [Hu05, Rio05, DDØ22], X is dualizable in the $\mathbb{A}^1$-stable homotopy category $\mathrm{SH}(k)$. It follows that an endomorphism $\varphi : X \to X$ has a trace $\mathrm{Tr}(\varphi)$ valued in the endomorphisms of the motivic sphere spectrum. A theorem of Morel [Mor04] identifies this endomorphism ring with the Grothendieck–Witt group $\mathrm{GW}(k)$ of k, defined to be the group completion of isomorphism classes of symmetric nondegenerate bilinear forms on k (see Section 2.2, and note that by Hoyois [Hoy15, Footnote 1], we need not assume k is perfect). We may therefore define the following enrichment to $\mathrm{GW}(k)$ of the logarithmic derivative of the zeta function.

Definition 1.1. Let X be a smooth, proper variety over a field k. Let $\varphi : X \to X$ be an endomorphism. The $\mathbb{A}^1$-logarithmic zeta function of (X, φ) is defined by

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi} := \sum_{m\geq 1} \mathrm{Tr}(\varphi^m) t^{m-1} \in \mathrm{GW}(k)[[t]].$$

In fact, the variety X need not be proper. By [LYZR19, Corollary B.2], a smooth scheme X over a field k is dualizable in the localized $\mathbb{A}^1$-stable homotopy category $\mathrm{SH}(k)_{\mathbb{Z}[\frac{1}{p}]}$ where p is the characteristic exponent of k. For an endomorphism $\varphi : X \to X$ of a smooth scheme over a field k, we have the more general definition

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi} := \sum_{m\geq 1} \mathrm{Tr}(\varphi^m) t^{m-1} \in \mathrm{GW}(k)[\tfrac{1}{p}][[t]].$$

For p odd, $\mathrm{GW}(k) \subseteq \mathrm{GW}(k)[\frac{1}{p}]$.

This $\mathbb{A}^1$-logarithmic zeta function recovers the classical zeta function via the rank map. In particular, $\mathbb{A}^1$-homotopy theory admits realization functors, including a symmetric monoidal stable étale realization $r_{\text{ét},\ell}$ from $\mathrm{SH}(k)$ to the derived category of ℓ-adic étale sheaves on the big étale site of k. It follows that $r_{\text{ét},\ell}(\mathrm{Tr}\,\varphi) = \mathrm{Tr}(r_{\text{ét},\ell}\varphi)$, which is the integer-valued trace of the usual Weil conjectures. The rank homomorphism, denoted $\mathrm{rank} : \mathrm{GW}(k) \to \mathbb{Z}$, sends the isomorphism class of a bilinear form $\beta : V \times V \to k$ to the dimension of the k-vector space V. Via the identification of $\mathrm{GW}(k)$ and the endomorphisms of the sphere spectrum, the étale realization map $r_{\text{ét},\ell} : \mathrm{End}(1_k) \to \mathrm{End}(r_{\text{ét},\ell}(1_k))$ is identified with rank. It follows that

$$\text{rank dlog}\, \zeta^{\mathbb{A}^1}_{X,\varphi} = \frac{d}{dt} \log \zeta_{X,\varphi}, \tag{1}$$

where $\zeta_{X,\varphi}$ denotes the classical zeta function

$$\zeta_{X,\varphi}(t) = \prod_i (P_{\varphi|H^i_{\text{ét}}}(t))^{(-1)^{i+1}} \qquad \text{where} \quad P_{\varphi|H^i_{\text{ét}}}(t) := \det(1 - t\varphi|H^i_{\text{ét}}(X_{k^s}; \mathbb{Z}_\ell)). \tag{2}$$

We investigate the additional information recorded in the $\mathbb{A}^1$-logarithmic zeta function. A case of particular interest in this paper is when k is finite and $\varphi : X \to X$ is the Frobenius endomorphism. If k is the finite field $\mathbb{F}_q$ of q elements, the Grothendieck–Witt group $\mathrm{GW}(\mathbb{F}_q)$ is computed as

$$\mathrm{GW}(\mathbb{F}_q) \cong \frac{\mathbb{Z}[\langle u \rangle]}{(\langle u \rangle^2 - 1, 2(\langle u \rangle - 1))}$$

where u is a fixed non-square in $\mathbb{F}_q$ and $\langle u \rangle$ denotes the class of the bilinear form $k \times k \to k$ sending (x, y) to uxy. As a group $\mathrm{GW}(\mathbb{F}_q)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, by sending a class in $\mathrm{GW}(k)$ to the pair given by its rank and its discriminant. As above, the rank gives the classical zeta function. The discriminant term can be computed in terms of $|X(\mathbb{F}_{q^m})|$ with Hoyois's beautiful enriched Grothendieck–Lefschetz trace formula [Hoy15] (see Section 8.2):

$$\text{disc dlog}\, \zeta^{\mathbb{A}^1}_{X,\varphi} = \sum_{m \geq 1} \left(\sum_{\substack{i|m \\ i \text{ even}}} \frac{1}{i} \sum_{d|i} \mu(d) |X(\mathbb{F}_{q^{i/d}})| \right) t^{m-1},$$

where μ denotes the Möbius function. Combining rank and discriminant yields the expression

$$\text{dlog}\, \zeta^{\mathbb{A}^1}_{X,\varphi} = \sum_{m \geq 1} \left(\sum_{i|m} \alpha(i)\, \mathrm{Tr}_{\mathbb{F}_{q^i}/\mathbb{F}_q} \langle 1 \rangle \right) t^{m-1}, \tag{3}$$

where $\alpha(i)$ denotes the number of points of X with residue field $\mathbb{F}_{q^i}$ and $\mathrm{Tr}_{\mathbb{F}_{q^i}/\mathbb{F}_q}\langle 1 \rangle$ is the transfer on GW (see (18) and (21) for the definition and computation of the transfer). Theorem 8.9 gives a further computation of $\text{dlog}\, \zeta^{\mathbb{A}^1}_{X,\varphi}$ for the Frobenius φ of X over a finite field in terms of point counts. It is amenable to (computer) computation up to finitely many coefficients of t^m, e.g., for elliptic curves; see Section 8.3. Finitely many coefficients of t^m determine the entire $\mathbb{A}^1$-logarithmic zeta function because the same is true for the classical zeta function.

Connections with topology and real points. Zeta functions gain power by relating the integers $|X(\mathbb{F}_{q^m})|$ to étale cohomology. The rationality result of the Weil conjectures (2) shows that the infinitely many integers $|X(\mathbb{F}_{q^m})|$ for $m = 1, 2, \ldots$ are determined by the coefficients of the $P_{\varphi|H^i_{\text{ét}}}(t)$, which are finite in number and admit a beautiful relationship with topology. Transcendental results relate étale cohomology to the complex points $\mathcal{X}(\mathbb{C})$ of a lift $\mathcal{X}$ of X to characteristic 0. The ranks of the singular cohomology groups of $\mathcal{X}(\mathbb{C})$ then determine the degrees of the $P_{\varphi|H^i_{\text{ét}}}(t)$. In this way, the classical zeta function is an algebraic manipulation of the numbers $|X(\mathbb{F}_{q^m})|$ related to the topology of $\mathcal{X}(\mathbb{C})$.

The $\mathbb{A}^1$-logarithmic zeta function (3) is similarly an algebraic manipulation of the $|X(\mathbb{F}_{q^m})|$, but it is related both to the topology of $\mathcal{X}(\mathbb{C})$ and to the topology of $\mathcal{X}(\mathbb{R})$ by the rank and discriminant, respectively. For example, the scheme $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{R}$ has a twist $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1$, given by the Weil restriction of scalars. The topology of the complex points of $\mathbb{P}^1 \times \mathbb{P}^1$ is unchanged by the twist, but the topology of the real points is changed from a product of two circles to a 2-sphere

$$(\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{R}) \simeq S^1 \times S^1 \qquad \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1(\mathbb{R}) \simeq S^2.$$

If q is a prime congruent to 3 modulo 4, then the extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ is given by $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{-1}]$ and the $\mathbb{R}$-schemes $\mathbb{P}^1 \times \mathbb{P}^1$ and $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1$ are lifts to characteristic zero of the varieties $\mathbb{P}^1 \times \mathbb{P}^1$ and $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ over $\mathbb{F}_q$, respectively. The change in the topology of the real points of the lifts to characteristic zero is reflected in the $\mathbb{A}^1$-zeta functions:

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{P}^1\times\mathbb{P}^1,\varphi} = \frac{d}{dt} \log \frac{1}{(1-t)(1-q_\epsilon^2 t)} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{(1-q_\epsilon t)^2}$$

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1,\varphi} = \frac{d}{dt} \log \frac{1}{(1-t)(1-q_\epsilon^2 t)} + \langle -u \rangle \frac{d}{dt} \log \frac{1}{1-q_\epsilon \langle u \rangle t} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{1+q_\epsilon t},$$

Here u is a nonsquare in $\mathbb{F}_q^*$, which we may take to be $u = -1$ under our assumption that $q \equiv 3 \pmod 4$. (See Examples 8.7 and 8.13 for details of the computation.) Applying the discriminant homomorphism $\operatorname{disc}\colon \operatorname{GW}(\mathbb{F}_q) \to \mathbb{F}_q^*/(\mathbb{F}_q^*)^2 \cong \mathbb{Z}/2\mathbb{Z}$ also gives different values:

$$\operatorname{disc} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{P}^1\times\mathbb{P}^1,\varphi} = \operatorname{disc}\langle -1 \rangle \frac{d}{dt} \log(1-t)^2 \neq 0,$$

$$\operatorname{disc} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1,\varphi} = 0.$$

See also Remark 8.15.

More generally, we prove results on the relationship between $\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi}$ and $\mathcal{X}(\mathbb{R})$ in Section 7. There is a signature homomorphism $\operatorname{sign} : \operatorname{GW}(\mathbb{R}) \to \mathbb{Z}$ that sends $a\langle 1 \rangle + b\langle -1 \rangle$ to $a - b$. In the presence of a lift of Frobenius, Proposition 7.3 and Corollary 7.4 say that in an appropriate sense, there is a well-defined signature of $\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\mathbb{F}_p}$ that can be computed as

$$\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\mathbb{F}_q} = \prod \det(1 - t\varphi(\mathbb{R}) | H^i_{\mathrm{top}}(\mathcal{X}(\mathbb{R}); \mathbb{Z})).$$

For example, this computes $\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi}$ for toric varieties; see Example 7.5. To eliminate the assumption that a lift of Frobenius exists, we turn to further machinery.

Trace formula and rationality. Missing from the discussion so far is an expression for $\operatorname{Tr}(\varphi)$ in terms of a trace on cohomology, analogous to the Grothendieck–Lefschetz trace formula

$$\operatorname{Tr}(\varphi^m) = \sum_{i=0}^{2d} (-1)^i \operatorname{Tr}(\varphi^m | H^i_{\acute{e}t}(X_{k^s}; \mathbb{Z}_\ell)).$$

Such formulas relate the classical zeta function of a variety over a finite field to the complex points of a lift of the variety, without requiring the much more restrictive hypothesis of a lift of Frobenius.

For this, we use Morel and Sawant's $\mathbb{A}^1$-cellular homology [MS20], currently defined for so-called cellular schemes, which are a generalization of the classical notion of varieties with an affine stratification. In [MS20, Remark 2.44], they note the existence of an extension of this theory to a pro-object for all spaces in the sense of $\mathbb{A}^1$-homotopy theory and in particular, for all smooth schemes over a field. Supported by conjectures on Poincaré duality for their theory, they conjecture that for smooth, projective varieties over a field, this pro-object is in fact constant [Mor22]. If their conjectures are true, our work may likewise extend to varieties which are not cellular. In this paper, we proceed for cellular schemes; see Section 3.1 for the definitions.

We use the machinery of Morel and Sawant to give a $\mathrm{GW}(k)$-enriched Grothendieck–Lefschetz trace formula:

Theorem 1. *Let $\varphi\colon X \to X$ be an endomorphism of a smooth projective simple cellular scheme X over a field k. In $\mathrm{GW}(k)$ we have the equality*

$$\mathrm{Tr}(\varphi) = \sum_i \langle -1\rangle^i \,\mathrm{Tr}(C_i^{\mathrm{cell}}(\varphi))$$

where $C_i^{\mathrm{cell}}(\varphi)$ denotes the $\mathbb{A}^1$-cellular complex of Morel and Sawant in degree i.

See Theorem 5.9. (This is a different result from the main theorem of [Hoy15] despite the similarity in the name.) This expression for $\mathrm{Tr}(\varphi)$ in terms of the trace of the cellular complex leads to rationality results for the $\mathbb{A}^1$-logarithmic zeta function, which we now describe.

Because we are working with the logarithmic derivative, we need an appropriate definition of rationality in this context. Here, we enrich the logarithmic derivative of the zeta function, rather than the zeta function itself, because the Grothendieck–Witt group in general has torsion elements. (See Remark 6.3 for why this does not lift to a notion of rationality using a standard λ-ring structure.)

Definition 1.2. Let R be a ring. We say a power series $\Phi(t) \in R[[t]]$ is dlog rational if there exists a finite collection of polynomials $P_j \equiv 1 \pmod{t}$ in $R[t]$ and elements $c_j \in R$ such that

$$\Phi(t) = \sum_j c_j \frac{P_j'(t)}{P_j(t)},$$

where $\frac{1}{P_j(t)} := \sum_{m\geq 0}(1-P_j(t))^m$.

Example 1.3. If a rational power series Ψ with integer coefficients can be put in the form

$$\Psi(t) = \prod_j P_j(t)^{c_j}$$

where $P_j(t)$ are polynomials with integer coefficients such that $P_j \equiv 1 \mod t$, and $c_j \in \mathbb{Z}$, then $\frac{d}{dt}\log\psi$ is dlog rational. In particular, the Weil conjectures imply that $\frac{d}{dt}\log\zeta_X(t)$ is dlog rational for every smooth projective variety X over a finite field.

We prove that the $\mathbb{A}^1$-logarithmic zeta functions of simple cellular schemes are dlog rational with polynomial terms coming from the characteristic polynomials of matrices of elements of $\mathrm{GW}(k)$ giving the action of the endomorphism ϕ (see Theorem 6.2):

Theorem 2. *Let k be a field, and let X be a smooth projective scheme over k with a simple cellular structure. Let $\varphi : X \to X$ be an endomorphism of X. The function $\mathrm{dlog}\,\zeta^{\mathbb{A}^1}_{X,\phi}$ is dlog rational. More precisely,*

$$\mathrm{dlog}\,\zeta^{\mathbb{A}^1}_{X,\varphi} = \sum_{i=-\infty}^{\infty} -\langle -1\rangle^i \frac{d}{dt}\log P_{C_i^{\mathrm{cell}}(\varphi)}(t), \quad \textit{where} \quad P_{C_i^{\mathrm{cell}}(\varphi)}(t) = \det(1 - tC_i^{\mathrm{cell}}(\varphi))$$

and $C_i^{\mathrm{cell}}(\varphi)$ is a square matrix of elements of $\mathrm{GW}(k)$.

Our methods also yield variations on such a result. For example, we weaken simple cellular structure to cellular structure in Theorem 6.1: we lose matrices of elements of $\mathrm{GW}(k)$ and dlog rationality in the sense of Definition 1.2 (at least in so far as we can currently prove!) in exchange for an abstract logarithmic derivative of a characteristic polynomial on the ith term of Morel–Sawant's cellular complex. This abstract logarithmic derivative of a characteristic polynomial is introduced in Definition 5.4 and is an abstraction of the elementary algebraic lemma that says that if $\varphi : V \to V$ is an endomorphism of a finite-dimensional vector space with characteristic polynomial $P_\varphi(t)$, then

$$\frac{d}{dt}\log P_\varphi(t) = -\sum_{m\geq 1} \mathrm{Tr}(\varphi^m)t^{m-1}.$$

We use the notational convention that $\frac{d}{dt}\log$ denotes taking a derivative of a logarithm, while dlog denotes a formal substitute.

In this setup, the logarithmic $\mathbb{A}^1$-zeta function still retains terms associated to the cellular complex. For example, in the logarithmic zeta functions for both $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ with the Frobenius φ, we have terms $\frac{d}{dt}\log((1-t)^{-1}(1-q_\epsilon^2 t)^{-1})$, which come from the degree 0 and 2 terms in the cellular complex (which up to quasi-isomorphism is independent of the cellular structure! [MS20, Corollary 2.42]). The difference comes from terms in degree 1. For $\mathbb{P}^1 \times \mathbb{P}^1$ these are straightforward to compute (see Corollary 8.2 for the computation of the $\mathbb{A}^1$-logarithmic zeta function of any strictly cellular, smooth, projective variety, including products of projective spaces and Grassmannians). We obtain a quadratic term $\langle -1\rangle \frac{d}{dt}\log \frac{1}{(1-q_\epsilon t)^2}$ from the two obvious 1-cells of $\mathbb{P}^1 \times \mathbb{P}^1$. The $\langle -1\rangle$ in front corresponds to the fact that the real dimension of these cells is odd. In $\mathrm{dlog}\, \zeta^{\mathbb{A}^1}_{\mathrm{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1, \varphi}$ by contrast, the contribution from the 1-cells is

$$\langle -u\rangle \frac{d}{dt}\log \frac{1}{(1-q_\epsilon\langle u\rangle t)} + \langle -1\rangle \frac{d}{dt}\log\frac{1}{(1+q_\epsilon t)},$$

which has vanishing discriminant, related to the disappearance of the 1-cells over $\mathbb{R}$.

Outline of paper. Section 2 introduces some background needed for the remainder of the paper. Section 3 constructs the categories of cellular schemes used in the paper. In Section 4, we use Morel–Sawant's [MS20] $\mathbb{A}^1$-cellular homology to define a symmetric monoidal functor $\tilde{C}^{\mathrm{cell}}_*$ from an $\mathbb{A}^1$-homotopy category of cellular schemes to a derived category of strictly $\mathbb{A}^1$-invariant sheaves. In Section 5, we pass to appropriate Spanier–Whitehead categories to obtain dualizable objects. This defines a symmetric monoidal functor $C^{\text{SW-cell}}_*$ from the $\mathbb{A}^1$-cellular-Spanier–Whitehead category to a $\underline{K}^{\mathrm{MW}}_1[1]$-Spanier–Whitehead category of a derived category of strictly $\mathbb{A}^1$-invariant sheaves. We give an explicit computation of the dual of $C^{\text{SW-cell}}_*(X_+)$ for X a scheme with a simple (e.g., strict) cellular structure; see Propositions 5.7 and 5.8. In Section 5.3, we show the resulting Grothendieck–Lefschetz trace formula. We then prove in Section 6 the desired rationality results by computing the trace using $C^{\text{SW-cell}}_*$. Some aspects of the relationship with real points is discussed in Section 7. In Section 8, we compute examples. Finally, in Section 9, we explore the link between our $\mathbb{A}^1$-logarithmic zeta function for the Frobenius with Kapranov's motivic zeta function, explaining that they appear to be different. We also show the $\mathbb{A}^1$-logarithmic zeta function defines a motivic measure on the (modified) Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{F}_q})$.

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2. Preliminaries

2.1. **Categorical preliminaries.** Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}, \tau_{\mathcal{C}})$ be a symmetric monoidal category in the sense of, e.g., [Mar09, 1.2].

Definition 2.1. (See, for example, [PS14, Definition 2.1].) An object A of a symmetric monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}, \tau_{\mathcal{C}})$ is dualizable if there is a dual object $\mathbb{D}A$ in $\mathcal{C}$ in the sense that there exist coevaluation and evaluation maps

$$\eta : 1_{\mathcal{C}} \to A \otimes_{\mathcal{C}} \mathbb{D}A \qquad \epsilon : \mathbb{D}A \otimes_{\mathcal{C}} A \to 1_{\mathcal{C}},$$

respectively, such that the composites

$$A \xrightarrow{\eta \otimes 1_A} A \otimes_{\mathcal{C}} \mathbb{D}A \otimes_{\mathcal{C}} A \xrightarrow{1_A \otimes \epsilon} A$$

$$\mathbb{D}A \xrightarrow{1_{\mathbb{D}A} \otimes \eta} \mathbb{D}A \otimes_{\mathcal{C}} A \otimes_{\mathcal{C}} \mathbb{D}A \xrightarrow{\epsilon \otimes 1_{\mathbb{D}A}} \mathbb{D}A$$

are the identity maps 1_A and $1_{\mathbb{D}A}$, respectively.

Definition 2.2. (See, for example, [PS14, Definition 2.2].) For an endomorphism $\varphi : A \to A$ of a dualizable object A with dual $\mathbb{D}A$, the categorical trace $\mathrm{tr}(\varphi) \in \mathrm{End}(1_{\mathcal{C}})$ is the composition

$$1_{\mathcal{C}} \xrightarrow{\eta} A \otimes_{\mathcal{C}} \mathbb{D}A \xrightarrow{\varphi \otimes_{\mathcal{C}} \mathrm{Id}_{\mathbb{D}A}} A \otimes_{\mathcal{C}} \mathbb{D}A \xrightarrow{\tau_{\mathcal{C}}} \mathbb{D}A \otimes_{\mathcal{C}} A \xrightarrow{\epsilon} 1_{\mathcal{C}}.$$

Definition 2.3. A symmetric monoidal functor is a functor $F : \mathcal{C} \to \mathcal{D}$ between symmetric monoidal categories, together with a natural isomorphism $F(1_{\mathcal{C}}) \cong 1_{\mathcal{D}}$ and for all objects $A, B \in \mathcal{C}$, natural isomorphisms $\iota_{A,B} : F(A) \otimes_{\mathcal{D}} F(B) \xrightarrow{\cong} F(A \otimes_{\mathcal{C}} B)$ satisfying associativity, unitality, and symmetry; the last is the condition that the diagrams

$$\begin{array}{ccc} F(A \otimes_{\mathcal{C}} B) & \xrightarrow{F(\tau_{\mathcal{C}})} & F(B \otimes_{\mathcal{C}} A) \\ \uparrow{\scriptstyle \iota_{A,B}} & & \uparrow{\scriptstyle \iota_{B,A}} \\ F(A) \otimes_{\mathcal{D}} F(B) & \xrightarrow{\tau_{\mathcal{D}}} & F(B) \otimes_{\mathcal{D}} F(A) \end{array}$$

commute (see [Mac71, Chapter XI]).

Proposition 2.4. *Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Let A be a dualizable object of $\mathcal{C}$ with dual $\mathbb{D}A$. Then*

(1) *$F(A)$ is dualizable with dual $F(\mathbb{D}A)$, and*
(2) *for any endomorphism $\varphi : A \to A$, we have $\mathrm{tr}(F(\varphi)) = F(\mathrm{tr}(\varphi))$.*

Proof. Straightforward. □

We recall the notion of a Spanier–Whitehead category [Voe98, Section 4].

Definition 2.5. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}, \tau_{\mathcal{C}})$ be a symmetric monoidal category and T be an object of $\mathcal{C}$. The Spanier–Whitehead category $\mathcal{C}[T^{\otimes -1}]$ is the category with objects (C, n), where C is an object of $\mathcal{C}$ and $n \in \mathbb{Z}$, and morphisms given by

$$\mathrm{Mor}_{\mathcal{C}[T^{\otimes -1}]}((C,n),(C',n')) = \operatorname*{colim}_{m \geq -n,-n'} \mathrm{Mor}_{\mathcal{C}}(T^{\otimes m+n} \otimes C, T^{\otimes m+n'} \otimes C')$$

Composition of morphisms is given by the expected formula.

By [Voe98, Theorem 4.3], the composition $(C, n) \otimes (C', n') := (C \otimes C', n + n')$ defines a symmetric monoidal structure on $\mathcal{C}[T^{\otimes -1}]$ provided that the cyclic permutation of $T \otimes T \otimes T$ is the identity. Moreover, in this case, the functor $\mathcal{C} \to \mathcal{C}[T^{\otimes -1}]$ sending C to $(C, 0)$ is symmetric monoidal and the object $(T, 0)$ has tensor inverse $T^{\otimes -1} = (1_{\mathcal{C}}, -1)$.

Proposition 2.6. *Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}}, \tau_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}}, \tau_{\mathcal{D}})$ be a symmetric monoidal categories and let T be an object of $\mathcal{C}$ such that the cyclic permutation of $T \otimes T \otimes T$ is the identity. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor. Then there is a unique (up to unique isomorphism) symmetric monoidal functor $F[T^{\otimes -1}] : \mathcal{C}[T^{\otimes -1}] \to \mathcal{D}[F(T)^{\otimes -1}]$ such that the diagram*

(4)
$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}[T^{\otimes -1}] & \xrightarrow[F[T^{\otimes -1}]]{} & \mathcal{D}[F(T)^{\otimes -1}] \end{array}$$

commutes.

Proof. Since the cyclic permutation of $T \otimes T \otimes T$ is the identity and F is a symmetric monoidal functor, the cyclic permutation of $F(T) \otimes F(T) \otimes F(T)$ is the identity. Thus $\mathcal{D}[F(T)^{\otimes -1}]$ is a symmetric monoidal category receiving a canonical symmetric monoidal functor from $\mathcal{D}$. Define $F[T^{\otimes -1}](C, n) := (F(C), n)$. Then $F[T^{\otimes -1}]$ is a symmetric monoidal functor such that (4) commutes. Uniqueness follows from the isomorphism $(C, n) \cong C \otimes_{\mathcal{C}[T^{\otimes -1}]} T^{\otimes n}$ in $\mathcal{C}[T^{\otimes -1}]$. Here C and T also denote their images $(C, 0)$ and $(T, 0)$, respectively, under the canonical functor $\mathcal{C} \to \mathcal{C}[T^{\otimes -1}]$. □

2.2. **On the motivic Spanier–Whitehead category and Milnor–Witt K-theory.** Let $\mathbf{Sm}_k$ denote the category of smooth schemes over a field k. We will be working with the Morel–Voevodsky $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ over k, and its pointed version $\mathcal{H}(k)_*$ [MV99, p. 109]. A feature of $\mathbb{A}^1$-homotopy theory is two different analogues of the circle: S^1 and $\mathbb{A}^1 - \{0\}$. We use the following indexing convention for the resulting spheres: let $S^{1,0} = S^1$, $S^{1,1} = \mathbb{A}^1 - \{0\}$, and $S^{p,q} = (S^{1,0})^{\wedge(p-q)} \wedge (S^{1,1})^{\wedge q}$. Let $\mathbb{P}^1$ denote projective space over k of dimension 1 pointed at ∞. There is a weak equivalence $\mathbb{P}^1 \simeq S^{2,1}$ in $\mathcal{H}(k)_*$ induced from the pushout and homotopy pushout $\mathbb{P}^1 \simeq \mathbb{A}^1 \cup_{\mathbb{G}_m} \mathbb{A}^1$. Inductive and gluing arguments show that $\mathbf{A}^n - \{0\} \simeq S^{2n-1,n}$ and $\mathbf{A}^n/\mathbf{A}^n - \{0\} \simeq S^{2n,n}$ [MV99, Example 2.20]. By [Mor12, Lemma 3.43(2)] (for example), the cyclic permutation on $\mathbb{P}^1 \wedge \mathbb{P}^1 \wedge \mathbb{P}^1$ is the identity in $\mathcal{H}(k)_*$. Let

$$\mathrm{SW}(k) := \mathcal{H}(k)_*[(\mathbb{P}^1)^{\otimes -1}]$$

be the Spanier–Whitehead category arising from $\mathcal{H}(k)_*$ with chosen object $\mathbb{P}^1$ (recall Definition 2.5); note that it is a symmetric monoidal category.

Given a vector bundle V on a smooth scheme X, the associated Thom space [MV99, p. 110-114] is defined as $\mathrm{Th}_X V := V/(V - 0)$, where $V - 0$ denotes the complement of the zero section. The stable $\mathbb{A}^1$-homotopy category $\mathrm{SH}(k)$ receives a symmetric monoidal functor from $\mathrm{SW}(k)$ which is fully faithful on smooth schemes and their Thom spaces [Voe98, Theorem 5.2, Corollary 5.3].

Fundamental theorems of Morel [Mor04, Mor12] compute certain stable and unstable homotopy groups of spheres in terms of Milnor–Witt K-theory. Let $K_*^{\mathrm{MW}}(k) = \oplus_{i=-\infty}^{\infty} K_i^{\mathrm{MW}}(k)$ denote the Milnor–Witt K-theory of a field k, defined by Morel and Hopkins to be the associative algebra generated by a symbol η of degree -1 and symbols $[u]$ for $u \in k^*$ of degree $+1$, subject to the relations

$$[u][1-u] = 0, \qquad [uv] = [u] + [v] + \eta[u][v], \qquad \eta[u] = [u]\eta, \qquad \eta h = 0$$

for all u,v in k^*, where $h = 2 + [-1]\eta$ denotes the hyperbolic element. Let $\underline{K}_*^{\mathrm{MW}}$ denote the associated unramified sheaf on $\mathbf{Sm}_k$. (See [Mor12, Chapter 3.2] for the definition and more information on the associated unramified sheaf.) Then Morel [Mor12, Corollary 6.43] shows that

$$\mathrm{Mor}_{\mathrm{SH}(k)}(S^n, S^{n,m}) \cong \underline{K}_m^{\mathrm{MW}} \text{for all } n, m \qquad \mathrm{Mor}_{\mathcal{H}(k)_*}(S^n, S^{n+m,m}) \cong \underline{K}_m^{\mathrm{MW}}, n \geq 2, m \geq 0.$$

The 0th graded piece $K_0^{\mathrm{MW}}(k)$ is isomorphic to the Grothendieck–Witt group $\mathrm{GW}(k)$ [Mor12, Lemma 3.10], defined to be the group completion of the semi-ring of nondegenerate symmetric bilinear forms. There is a presentation of $\mathrm{GW}(k)$ with generators

$$\begin{array}{rccc} \langle a \rangle : & k \times k & \to & k \\ & (x, y) & \mapsto & axy \end{array}$$

for every $a \in k^*$, and relations given for all $a, b \in k^*$ by:

(1) $\langle ab^2 \rangle = \langle a \rangle$;
(2) $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$
(3) $\langle a \rangle \langle b \rangle = \langle ab \rangle$.

The sheaf $\underline{K}_0^{\mathrm{MW}}$, also denoted $\underline{\mathrm{GW}}$, is the Nisnevich sheaf associated to the presheaf sending a smooth k-scheme Y to the group completion of the semi-ring of isomorphism classes of locally free sheaves V on Y equipped with a non-degenerate symmetric bilinear form $V \times V \to \mathcal{O}_Y$.

Let $\mathbf{Ab}(k)$ denote the abelian category of Nisnevich sheaves of abelian groups on $\mathbf{Sm}_k$. For future reference, we record the following well-known fact.

Lemma 2.7. *There exists a natural isomorphism* $\underline{K}_m^{\mathrm{MW}} \to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(\underline{K}_n^{\mathrm{MW}}, \underline{K}_{n+m}^{\mathrm{MW}})$ *for all* $n, m \geq 0$.

Proof. See also [Mor12, 3.2]. The natural map is multiplication on $\underline{K}_*^{\mathrm{MW}}$, or equivalently, that of [Mor12, Lemma 3.49]. Let $(-)_{-1}$ denote the -1 construction of Voevodsky [Mor12, p 33]. Applying this construction n times produces a map $(-)_{-1}^{\circ n} : \mathrm{Hom}_{\mathbf{Ab}(k)}(\underline{K}_n^{\mathrm{MW}}, \underline{K}_{n+m}^{\mathrm{MW}}) \to$

$\mathrm{Hom}_{\mathbf{Ab}(k)}(\underline{K}_0^{MW}, \underline{K}_m^{MW})$. Evaluation at 1 in $\underline{K}_0^{MW}$ defines a map $\mathrm{Hom}_{\mathbf{Ab}(k)}(\underline{K}_0^{MW}, \underline{K}_m^{MW}) \to \underline{K}_m^{MW}$, inverse to the given natural map. $\square$

2.3. **$\mathbb{A}^1$-derived category and $\mathbb{A}^1$-homology.** Let $\mathbf{Ch}(\mathbf{Ab}(k))$ be the category of chain complexes C_* of Nisnevich sheaves of abelian groups on $\mathbf{Sm}_k$ with differentials of degree -1. We denote by $D(\mathbf{Ab}(k))$ the associated derived category, obtained by inverting quasi-isomorphisms [Mor12, 6.2]. Let $\mathbb{Z}(\mathbb{A}^1)$ denote the free sheaf of abelian groups on the sheaf of sets represented by $\mathbb{A}^1$. Note that the map $\mathbb{A}^1 \to \mathrm{Spec}\, k$ induces a map $\mathbb{Z}(\mathbb{A}^1) \to \mathbb{Z}$.

A chain complex C_* in $\mathbf{Ch}(\mathbf{Ab}(k))$ is defined to be $\mathbb{A}^1$-*local* if for any D_* in $\mathbf{Ch}(\mathbf{Ab}(k))$, the map

$$\mathrm{Hom}_{D(\mathbf{Ab}(k))}(D_*, C_*) \to \mathrm{Hom}_{D(\mathbf{Ab}(k))}(D_* \otimes \mathbb{Z}(\mathbb{A}^1), C_*)$$

is a bijection [Mor12, Definition 6.17]. The $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(\mathbf{Ab}(k))$ [Mor12, Definition 6.17] is obtained from $\mathbf{Ch}(\mathbf{Ab}(k))$ by inverting the $\mathbb{A}^1$-quasi-isomorphisms, defined to be morphisms $f : C_* \to D_*$ such that for all $\mathbb{A}^1$-local chain complexes E_*

$$\mathrm{Hom}_{D(\mathbf{Ab})}(D_*, E_*) \to \mathrm{Hom}_{D(\mathbf{Ab})}(C_*, E_*)$$

is bijective. There is an *(abelian) $\mathbb{A}^1$-localization functor* $L^{ab}_{\mathbb{A}^1} : \mathbf{Ch}(\mathbf{Ab}(k)) \to \mathbf{Ch}(\mathbf{Ab}(k))$ inducing a left-adjoint to the inclusion of $\mathbb{A}^1$-local complexes in $D(\mathbf{Ab}(k))$ [Mor12, 6.18, 6.19]; it induces an equivalence of categories between $D_{\mathbb{A}^1}(\mathbf{Ab}(k))$ and the full subcategory of $\mathbb{A}^1$-local complexes.

For a simplicial Nisnevich sheaf $\mathcal{X}$ on $\mathbf{Sm}_k$, let $C_*(\mathcal{X})$ in $\mathbf{Ch}(\mathbf{Ab}(k))$ denote the normalized chain complex associated to the free simplicial abelian group $\mathbb{Z}\mathcal{X}$ on $\mathcal{X}$. The $\mathbb{A}^1$-chain complex $C_*^{\mathbb{A}^1}(\mathcal{X})$ (see [Mor12, Chapter 6.2]) is obtained by taking the $\mathbb{A}^1$-localization of $C_*(\mathcal{X})$. This defines a functor from the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ to the derived category $D_{\mathbb{A}^1}(\mathbf{Ab}(k))$:

$$C_*^{\mathbb{A}^1} : \mathcal{H}(k) \to D_{\mathbb{A}^1}(\mathbf{Ab}(k)).$$

If $\mathcal{X}$ is a pointed space, the reduced chain complex $\widetilde{C}_*(\mathcal{X})$, which is the kernel of the morphism of $C_*(\mathcal{X})$ to $\mathbb{Z}$, similarly gives rise to a reduced $\mathbb{A}^1$-chain complex $\widetilde{C}_*^{\mathbb{A}^1}(\mathcal{X})$.

Definition 2.8. For any simplicial Nisnevich sheaf $\mathcal{X}$ on $\mathbf{Sm}_k$ and integer n, the nth $\mathbb{A}^1$-homology sheaf $H_n^{\mathbb{A}^1}(\mathcal{X})$ of $\mathcal{X}$ is the nth homology sheaf of the $\mathbb{A}^1$-chain complex $C_*^{\mathbb{A}^1}(\mathcal{X})$. For a pointed space $\mathcal{X}$, the nth reduced $\mathbb{A}^1$-homology sheaf of $\mathcal{X}$ is $\widetilde{H}_n^{\mathbb{A}^1}(\mathcal{X}) := H_n^{\mathbb{A}^1}(\widetilde{C}_*^{\mathbb{A}^1}(\mathcal{X}))$.

Morel shows that $\mathbb{A}^1$-homology sheaves are strictly $\mathbb{A}^1$-invariant [Mor12, Corollary 6.23] in the following sense.

Definition 2.9. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathbf{Sm}_k$ for the Nisnevich topology. Then $\mathcal{F}$ is $\mathbb{A}^1$-invariant if for every $U \in \mathbf{Sm}_k$, the projection map $U \times \mathbb{A}^1 \to U$ induces a bijection $\mathcal{F}(U) \to \mathcal{F}(U \times A^1)$. The sheaf $\mathcal{F}$ is said to be strictly $\mathbb{A}^1$-invariant if for every $U \in \mathbf{Sm}_k$ and every integer $i \geq 0$, the projection map $U \times \mathbb{A}^1 \to U$ induces a bijection

$$H^i_{Nis}(U, \mathcal{F}) \to H^i_{Nis}(U \times \mathbb{A}^1, \mathcal{F}).$$

The category of strictly $\mathbb{A}^1$-invariant sheaves on $\mathbf{Sm}_k$ is denoted $\mathbf{Ab}_{\mathbb{A}^1}(k)$.

For example, for a presheaf $\mathcal{X}$ of sets on $\mathbf{Sm}_k$, the 0th $\mathbb{A}^1$-homology $H_0^{\mathbb{A}^1}(\mathcal{X})$ is the free strictly $\mathbb{A}^1$-invariant sheaf on $\mathcal{X}$. Morel computes

$$\widetilde{H}_n^{\mathbb{A}^1}(\mathbf{A}^n/\mathbf{A}^n - \{0\}) \cong \underline{K}_n^{MW} \text{ for } n \geq 1. \tag{5}$$

This follows from the $\mathbb{A}^1$-weak equivalence $\mathbf{A}^n/\mathbf{A}^n - \{0\} \simeq (S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}$ [MV99, p. 110, Spheres, Suspensions, Thom Spaces], the suspension isomorphism $\widetilde{H}_n^{\mathbb{A}^1}((S^1)^{\wedge n} \wedge (\mathbb{G}_m)^{\wedge n}) \cong \widetilde{H}_{n-1}^{\mathbb{A}^1}((S^1)^{\wedge n-1} \wedge (\mathbb{G}_m)^{\wedge n})$ [Mor12, Remark 6.30] and $\widetilde{H}_0^{\mathbb{A}^1}((\mathbb{G}_m)^{\wedge n}) \cong \underline{K}_n^{\mathrm{MW}}$[Mor12, Theorem 3.37 and Theorem 5.46]. For $n = 0$, because $\mathbf{A}^n/\mathbf{A}^n - \{0\} \cong \operatorname{Spec} k_+$ and $\widetilde{H}_n^{\mathbb{A}^1}(\operatorname{Spec} k_+) \cong \mathbb{Z}$, we have

$$\widetilde{H}_n^{\mathbb{A}^1}(\mathbf{A}^n/\mathbf{A}^n - \{0\}) \cong \mathbb{Z} \text{ for } n = 0.$$

By [Mor12, Corollary 6.24], $\mathbf{Ab}_{\mathbb{A}^1}(k)$ is an abelian category and the inclusion $\mathbf{Ab}_{\mathbb{A}^1}(k) \subset \mathbf{Ab}(k)$ is exact. There is a tensor product, denoted $\otimes_{\mathbb{A}^1}$ or $\otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)}$, on $\mathbf{Ab}_{\mathbb{A}^1}(k)$, defined by $M \otimes_{\mathbb{A}^1} N := \pi_0 L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes_{\mathbf{Ab}(k)} N)$ for $N, M \in \mathbf{Ab}_{\mathbb{A}^1}(k)$. The map $M \otimes N \to M \otimes_{\mathbb{A}^1} N$ is the initial map to a strictly $\mathbb{A}^1$-invariant sheaf, as we now explain: by [Mor12, Theorem 6.22], $L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes_{\mathbf{Ab}(k)} N)$ is -1-connected, and by [Mor12, Corollary 6.23], $\pi_0 L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes_{\mathbf{Ab}(k)} N)$ is an element of $\mathbf{Ab}_{\mathbb{A}^1}(k)$. A strictly $\mathbb{A}^1$-invariant sheaf F, by [Mor12, Corollary 6.23], is $\mathbb{A}^1$-local. Then a map $M \otimes N \to F$ factors uniquely $M \otimes N \to L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes N) \to F$. Since F has no higher homotopy groups, we have that $M \otimes N \to F$ thus factors uniquely $M \otimes N \to L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes N) \to \pi_0 L_{\mathbb{A}^1}^{\mathrm{ab}}(M \otimes N) \to F$.

Definition 2.10. (See [MS20, Definition 2.9].) A scheme X in $\mathbf{Sm}_k$ is cohomologically trivial if for every $n \geq 1$ and every strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathbf{Ab}_{\mathbb{A}^1}(k)$, we have $H_{\mathrm{Nis}}^n(X, M) = 0$.

If X in $\mathbf{Sm}_k$ is cohomologically trivial, then $H_0^{\mathbb{A}^1}(X)$ is projective in $\mathbf{Ab}_{\mathbb{A}^1}(k)$ [MS20, Example 2.36]. Moreover, if X and Y are cohomologically trivial in $\mathbf{Sm}_k$, so is $X \times Y$, and we have $H_0^{\mathbb{A}^1}(X \times Y) \cong H_0^{\mathbb{A}^1}(X) \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} H_0^{\mathbb{A}^1}(Y)$ [MS20, Remark 2.10 and p. 12].

Let $D(\mathbf{Ab}_{\mathbb{A}^1}(k))$ denote the bounded derived category of $\mathbf{Ab}_{\mathbb{A}^1}(k)$, obtained from the category $\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))$ by inverting quasi-isomorphisms. Let $D(\mathbf{Ab}_{\mathbb{A}^1}^p(k)) \subseteq D(\mathbf{Ab}_{\mathbb{A}^1}(k))$ denote the full subcategory on bounded complexes C_* of strictly $\mathbb{A}^1$-invariant sheaves C_* that are isomorphic either to $H_0^{\mathbb{A}^1}(X)$ for some cohomologically trivial X in $\mathbf{Sm}_k$ or to $\underline{K}_n^{\mathrm{MW}} \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} H_0^{\mathbb{A}^1}(X)$ for some cohomologically trivial X and $n \geq 1$. In particular, the C_* are projective [MS20, Remark 2.26(1)].

We give $D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))$ the structure of a symmetric monoidal category as follows. For chain complexes P_* and P_*' representing objects of $D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))$, the set of homomorphisms $\mathrm{Hom}_{D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))}(P_*, P_*')$ is given by homotopy classes of chain maps. Moreover, because $\underline{K}_n^{\mathrm{MW}} \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} \underline{K}_m^{\mathrm{MW}} \cong \underline{K}_{n+m}^{\mathrm{MW}}$ for $n, m \geq 1$ [Mor12, Theorem 3.37] and $H_0^{\mathbb{A}^1}(X \times Y) \cong H_0^{\mathbb{A}^1}(X) \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} H_0^{\mathbb{A}^1}(Y)$ as above, the tensor product of bounded chain complexes in $\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))$ determines a well-defined derived tensor product

$$\begin{aligned} (6) \qquad & \otimes_{D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))} : D(\mathbf{Ab}_{\mathbb{A}^1}^p(k)) \times D(\mathbf{Ab}_{\mathbb{A}^1}^p(k)) \to D(\mathbf{Ab}_{\mathbb{A}^1}^p(k)) \\ & \qquad P_* \otimes_{D(\mathbf{Ab}_{\mathbb{A}^1}(k))} P_*' := P_* \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1})(k)} P_*' \\ & \qquad \text{with} \qquad d(x \otimes y) = d \otimes 1 + (-1)^{|x|} 1 \otimes d. \end{aligned}$$

The symmetry isomorphism is defined as

$$(7) \qquad \tau_{D(\mathbf{Ab}_{\mathbb{A}^1}(k))} : P_* \otimes_{D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))} P_*' \to P_*' \otimes_{D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))} P_*$$

so that its restriction to $P_i \otimes P_j'$ maps to $P_j' \otimes P_i$ by multiplication by $(-1)^{ij}$ composed with the swap isomorphism for $\otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)}$. The swap map (7) and the derived tensor product of chain complexes (6) give $D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))$ the structure of a symmetric monoidal category.

3. $\mathbb{A}^1$-Spanier–Whitehead category of cellular smooth schemes

In this section, we define cellular schemes, which are closely related to the classical notion of schemes with an affine stratification, and we introduce the cellular Spanier-Whitehead category that we will use for the rest of the paper.

3.1. Cellular schemes.

There are numerous useful definitions of a cellular scheme or cellular object of $SH(k)$ available in the literature; see, e.g.,[Ful84, Example 1.9.1], [EH16, Definition 1.16] [Tot14, Section 3], [Roz06, Def 3.4.1], [DI05, Definitions 2.1, 2.7, and 2.10], [MS20, Definitions 2.7 and 2.11], and [AFH20, Question 1]. We will use [MS20, Definition 2.11] and a slight modification adapted to our purposes.

The basic goal is to define a class of varieties that are built out of simple varieties (from a cohomological perspective) in such a way that their cohomology can be understood inductively. A first natural class of such varieties are those built out of affine spaces.

Definition 3.1. An n-dimensional smooth scheme X has a strict cellular structure if one of the following equivalent conditions holds:

(1) (cf. [Ful84, Example 1.9.1]) X admits a filtration by closed subschemes

$$\emptyset = \Sigma_{-1}(X) \subset \Sigma_0(X) \subset \Sigma_1(X) \subset \cdots \subset \Sigma_{n-1}(X) \subset \Sigma_n(X) = X,$$

such that $\Sigma_i(X) \smallsetminus \Sigma_{i-1}(X)$ is a (finite) disjoint union of schemes isomorphic to affine space $\mathbb{A}^i$.

(2) (cf. [MS20, Definition 2.7]) X admits a filtration by open subschemes

$$\emptyset = \Omega_{-1}(X) \subset \Omega_0(X) \subset \Omega_1(X) \subset \cdots \subset \Omega_{n-1}(X) \subset \Omega_n(X) = X,$$

such that $\Omega_i(X) \smallsetminus \Omega_{i-1}(X)$ is a (finite) disjoint union of schemes isomorphic to affine space $\mathbb{A}^{n-i}$.

To pass between these conditions, let $\Omega_i(X) := X \smallsetminus \Sigma_{n-i-1}(X)$ and $\Sigma_i(X) := X \smallsetminus \Omega_{n-i-1}(X)$. A scheme with a strict cellular structure is called a strict cellular scheme.

A related notion is an affine stratification [EH16, Definition 1.16], where X is a disjoint union of finitely many locally closed subschemes U_i ("open strata"), each isomorphic to $\mathbb{A}^{n_i}$, whose closures $\overline{U}_i$ are a disjoint union of strata U_j. If X has an affine stratification, then defining $\Sigma_j(X)$ to be the union of all open strata U_i of dimension at most j yields a strict cellular structure.

Example 3.2. Projective space $\mathbb{P}^n$ has a strict cellular structure given by a full flag $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \cdots \subset \mathbb{P}^n$ with $\Sigma_i(X) := \mathbb{P}^i$ and $\Omega_i(X) := \mathbb{P}^n \smallsetminus \mathbb{P}^{n-i-1}$. Since the join $\mathbb{P}^i * \mathbb{P}^j \simeq_{\mathbb{A}^1} \mathbb{P}^{i+j+1}$, it is useful to think of $\Omega_i(X)$ as an open neighborhood of the i-skeleton. See [MS20, Remark 2.12] for more discussion of the topological notion of skeleta, CW-structure, and cellular structure.

Example 3.3. The class of schemes with a strict cellular structure contains many geometrically interesting examples. Grassmannians $\mathbb{G}(r, n)$ have an affine stratification by the Schubert cells [EH16, Section 4.1.2]. More generally, the Bruhat decomposition on the flag vareity G/B of a split semisimple algebraic group G with a Borel subgroup B defines a strict cellular structure [MS20, p. 12].

The following more general definition of Morel–Sawant allows the "cells" to be more general affine schemes.

Definition 3.4. ([MS20, Definition 2.11]) Let X be a smooth scheme over a field k. A cellular structure on X consists of an increasing filtration

$$\emptyset = \Omega_{-1}(X) \subsetneq \Omega_0(X) \subsetneq \Omega_1(X) \subsetneq \cdots \subsetneq \Omega_s(X) = X$$

by open subschemes such that for $i \in \{0, \ldots, s\}$, the reduced induced closed subscheme $X_i := \Omega_i(X) \smallsetminus \Omega_{i-1}(X)$ of $\Omega_i(X)$ is k-smooth, affine, everywhere of codimension i and cohomologically trivial. A scheme with a cellular structure is called a cellular scheme.

Definition 3.5. A cellular structure on X is called simple if for every n, $H_0^{\mathbb{A}^1}(X_n) \simeq \mathbb{Z}^{b_n}$ for some non-negative integer b_n. A scheme with a simple cellular structure is called a simple cellular scheme.

The class of simple cellular schemes contains all schemes with a strict cellular structure (and hence with an affine stratification), since $H_0^{\mathbb{A}^1}(\mathbb{A}^i) \simeq \mathbb{Z}$ and $H^n_{\mathrm{Nis}}(\mathbb{A}^i, M) = 0$ for all $n > 0$ and any strictly $\mathbb{A}^1$-invariant sheaf $M \in \mathbf{Ab}_{\mathbb{A}^1}(k)$.

3.2. **Cellular Spanier–Whitehead category.** Let $\mathcal{H}(k)_*^{\mathrm{cell}}$ denote the full subcategory of $\mathcal{H}(k)_*$ with objects consisting of pointed spaces of the form $\mathrm{Th}_X V$, where X is a smooth k-scheme admitting a cellular structure and V is a vector bundle on X. Note that $X_+ := X \coprod \mathrm{Spec}\, k$ is an object of $\mathcal{H}(k)_*^{\mathrm{cell}}$ as $X_+ \cong \mathrm{Th}_{X_+} 0$. The product of two smooth schemes admitting cellular structures admits a canonical cellular structure (see [MS20, p.21] or Lemma 4.5). It follows that the symmetric monoidal structure $\wedge$ on $\mathcal{H}(k)_*$ restricts to a symmetric monoidal structure on $\mathcal{H}(k)_*^{\mathrm{cell}}$.

We now introduce the cellular Spanier–Whitehead category

$$\mathrm{SW}^{\mathrm{cell}}(k) := \mathcal{H}(k)_*^{\mathrm{cell}}[(\mathbb{P}^1)^{\otimes -1}].$$

Recall that we defined $\mathbb{P}^1$ to be the projective line pointed at ∞. The pointed space $\mathbb{P}^1$ is in $\mathcal{H}(k)_*^{\mathrm{cell}}$ because $\mathbb{P}^1 = \mathrm{Th}_{\mathrm{Spec}\, k}\, \mathcal{O}$, so $\mathrm{SW}^{\mathrm{cell}}(k)$ is well-defined. Note that the canonical functor $\mathrm{SW}^{\mathrm{cell}}(k) \to \mathrm{SW}(k)$ is fully faithful. As noted in Section 2.1, since the cyclic permutation on $\mathbb{P}^1$ is the identity, the natural functor $\mathcal{H}(k)_*^{\mathrm{cell}} \to \mathrm{SW}^{\mathrm{cell}}(k)$ is a symmetric monoidal functor.

There is a canonical $\mathbb{A}^1$-weak equivalence

$$\mathrm{Th}_X V \xleftarrow{\sim} \mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V),$$

between the Thom space $\mathrm{Th}_X V := V/(V - 0)$ and the quotient $\mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V)$, where $\mathcal{O}$ denotes the trivial bundle. It follows that there is a canonical isomorphism $\mathrm{Th}_X(V \oplus \mathcal{O}) \cong \mathbb{P}^1 \wedge \mathrm{Th}_X V$ in $\mathcal{H}(k)_*$ [MV99, Prop 2.17]. This can be used to extend the Thom space construction to a functor

$$\mathrm{Th} : K_0(X) \to \mathrm{SW}(k) \qquad \mathrm{Th}\, V := (\mathrm{Th}_X(V \oplus \mathcal{O}^r), -r)$$

where r denotes a positive integer such that $V \oplus \mathcal{O}^r$ is represented by a vector bundle. Indeed, a path in the K-theory groupoid $K(X)$ between vector bundles V and W produces a canonical isomorphism $\mathrm{Th}_X V \simeq \mathrm{Th}_X W$ in $\mathrm{SH}(k)$ [Ayo07, Scholie 1.4.2(2)]. This isomorphism lies in $\mathrm{SW}(k)$ because the functor $\mathrm{SW}(k) \to \mathrm{SH}(k)$ is fully faithful on Thom spaces. For V a vector bundle (of non-negative rank), there is a canonical isomorphism between the Thom space functor Th just defined applied to V and the usual $\mathrm{Th}_X V$, justifying the abuse of notation. When X admits a simple cellular structure, the Thom space functor Th factors through $\mathrm{SW}^{\mathrm{cell}}(k) \to \mathrm{SW}(k)$, defining $\mathrm{Th} : K_0(X) \to \mathrm{SW}^{\mathrm{cell}}(k)$.

Lemma 3.6. *Let X be a smooth projective scheme over k and suppose that X admits a cellular structure. Then X_+ is dualizable in $\mathrm{SW}^{\mathrm{cell}}(k)$.*

Proof. Since X is smooth and projective over k, we have that X_+ is dualizable in $\mathrm{SH}(k)$ with dual $\mathbb{D}X_+ \simeq \mathrm{Th}(-TX)$, where TX denotes the tangent bundle of X [Rio05, Théorème 2.2]. Let $\mathbb{D}X_+$ in $\mathrm{SW}^{\mathrm{cell}}(k)$ denote $\mathrm{Th}(-TX)$ by a slight abuse of notation. The evaluation and coevaluation maps ϵ and η in $\mathrm{SH}(k)$ of Definition 2.1 have unique preimages under $\mathrm{SW}^{\mathrm{cell}}(k) \to \mathrm{SW}(k) \to \mathrm{SH}(k)$ because both functors are fully faithful on $1_{\mathrm{SW}^{\mathrm{cell}}(k)}$, X_+, $\mathbb{D}X$, $X_+ \wedge \mathbb{D}X$, etc. □

4. The cellular homology of Morel–Sawant on cellular Thom spaces

Morel and Sawant [MS20, §2.3] define the cellular $\mathbb{A}^1$-chain complex and corresponding cellular $\mathbb{A}^1$-homology for a cellular scheme, and show these are functors between appropriate categories [MS20, Corollary 2.43] preserving monoidal structures [MS20, Lemma 2.31]. We generalize these definitions in a straightforward manner to include Thom spaces, which will result in a symmetric monoidal functor

$$\tilde{C}^{\mathrm{cell}}_* : \mathcal{H}(k)^{\mathrm{cell}}_* \to D(\mathbf{Ab}^{p}_{\mathbb{A}^1}(k))$$

(see Proposition 4.7).

Let V be a vector bundle of rank $r \geq 0$ on X in $\mathbf{Sm}_k$ and

$$\emptyset = \Omega_{-1} \subsetneq \Omega_0 \subsetneq \Omega_1 \subsetneq \cdots \subsetneq \Omega_s = X$$

be a cellular structure on X, so $\mathrm{Th}_X V$ is an object of $\mathrm{SW}^{\mathrm{cell}}(k)$. Let $X_i = \Omega_i \setminus \Omega_{i-1} = \coprod_{m\in M_i} X_{im}$ be the decomposition of X_i into connected components. Morel–Voevodsky Purity [MV99, Theorem 2.23] provides a canonical weak equivalence

$$\mathrm{Th}_{\Omega_i} V/\mathrm{Th}_{\Omega_{i-1}} V \simeq_{\mathbb{A}^1} \mathrm{Th}_{X_i}(V+\nu_i)$$

where ν_i is the normal bundle of the closed immersion $X_i \hookrightarrow \Omega_i$ of smooth k-schemes.

Consider the cofibration sequence

$$\mathrm{Th}_{\Omega_{i-1}} V \to \mathrm{Th}_{\Omega_i} V \to \mathrm{Th}_{\Omega_i} V/\mathrm{Th}_{\Omega_{i-1}} V,$$

and its long exact sequence of reduced $\mathbb{A}^1$-homology sheaves

$$\begin{array}{c} \cdots \to \widetilde{H}^{\mathbb{A}^1}_n(\mathrm{Th}_{\Omega_{i-1}} V) \to \widetilde{H}^{\mathbb{A}^1}_n(\mathrm{Th}_{\Omega_i} V) \to \widetilde{H}^{\mathbb{A}^1}_n(\mathrm{Th}_{\Omega_i} V/\mathrm{Th}_{\Omega_{i-1}} V) \to \widetilde{H}^{\mathbb{A}^1}_{n-1}(\mathrm{Th}_{\Omega_{i-1}} V) \to \cdots \\ \downarrow \simeq \\ \widetilde{H}^{\mathbb{A}^1}_n(\mathrm{Th}_{X_i}(V+\nu_i)) \end{array}$$

which gives the boundary map ∂_n as the composite

$$\widetilde{H}^{\mathbb{A}^1}_{n+r}(\mathrm{Th}_{X_n}(V+\nu_n)) \to \widetilde{H}^{\mathbb{A}^1}_{n+r-1}(\mathrm{Th}_{\Omega_{n-1}} V) \to \widetilde{H}^{\mathbb{A}^1}_{n+r-1}(\mathrm{Th}_{\Omega_n} V) \to \widetilde{H}^{\mathbb{A}^1}_{n+r-1}(\mathrm{Th}_{X_{n-1}}(V+\nu_{n-1})).$$

For $n \geq 0$, let

$$\tilde{C}^{\mathrm{cell}}_{n+r}(\mathrm{Th}_X V) := \tilde{C}^{\mathrm{cell}}_{n+r}(X,V) := \widetilde{H}^{\mathbb{A}^1}_{n+r}(\mathrm{Th}_{X_n}(V+\nu_n))$$

with boundary maps ∂_n as above be the (shifted, reduced) cellular $\mathbb{A}^1$-chain complex on (X,V). With this notation, we record the cellular structure of X also when we write $\mathrm{Th}_X V$. Note also that $\mathrm{Th}_X 0 \simeq X_+$ and $\tilde{C}^{\mathrm{cell}}_* \mathrm{Th}_X 0 \simeq C^{\mathrm{cell}}_* X$ with $C^{\mathrm{cell}}_* X$ defined by Morel and Sawant [MS20, Section 2.3].

Remark 4.1. Suppose X in $\mathbf{Sm}_k$ is equipped with a simple cellular structure. By definition, $\tilde{C}^{\mathrm{cell}}_n(\mathrm{Th}_X V) \cong \widetilde{H}^{\mathbb{A}^1}_n(\mathrm{Th}_{X_{n-r}}(V+\nu_n))$. By [MS20, Lemma 2.13], we may choose a trivialization of $V+\nu_n$. Such a choice defines an $\mathbb{A}^1$-weak equivalence $\mathrm{Th}_{X_{n-r}}(V+\nu_n) \simeq (X_{n-r})_+ \wedge (\mathbf{A}^n/\mathbf{A}^n - \{0\})$. By Morel's computation (5), we have $\widetilde{H}^{\mathbb{A}^1}_n((X_{n-r})_+ \wedge \mathbf{A}^n/\mathbf{A}^n - \{0\}) \cong H^{\mathbb{A}^1}_0(X_{n-r}) \otimes \underline{K}^{\mathrm{MW}}_n$

for $n \geq 1$ and $\widetilde{H}_n^{\mathbb{A}^1}((X_{n-r})_+ \wedge \mathbf{A}^n/\mathbf{A}^n - \{0\})) \cong H_0^{\mathbb{A}^1}(X_{n-r})$ for $n = 0$. Since the cellular structure on X is simple, $H_0^{\mathbb{A}^1}(X_i) \cong \mathbb{Z}^{b_i}$. Thus we have

$$\tilde{C}_i^{\mathrm{cell}}(\mathrm{Th}_X V) \cong \begin{cases} (\underline{K}_i^{\mathrm{MW}})^{b_{i-r}} & i > 0 \\ \mathbb{Z}^{b_{-r}} & i = 0. \end{cases}$$

If the cellular structure is additionally strict, then $\Omega_i \setminus \Omega_{i-1} \cong \coprod_{j=1}^{b_i} \mathbf{A}^{n-i}$. In other words, the integer b_i corresponds to the number of connected components of $\Omega_i \setminus \Omega_{i-1}$.

Remark 4.2. More generally, for X in $\mathbf{Sm}_k$ equipped with a cellular structure, $\tilde{C}_n^{\mathrm{cell}}(\mathrm{Th}_X V) \cong H_0^{\mathbb{A}^1}(X_{n-r}) \otimes \underline{K}_n^{\mathrm{MW}}$. Thus $\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V)$ is in $D(\mathbf{Ab}_{\mathbb{A}^1}^p(k))$.

To show functoriality of $\tilde{C}_*^{\mathrm{cell}}$, Morel and Sawant introduce the notion of a strict $\mathbb{A}^1$-resolution [MS20, Definition 2.33] and we will need this notion as well.

Proposition 4.3. *Let* $\mathrm{Th}_X V$ *be an object of* $\mathcal{H}_*^{\mathrm{cell}}(k)$. *There exists a unique morphism*

$$\phi_X : \tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_X V) \to \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V)$$

in $D(\mathbf{Ab}_{\mathbb{A}^1}(k))$ *that is a strict* $\mathbb{A}^1$*-resolution in the sense that the functor*

$$D(\mathbf{Ab}_{\mathbb{A}^1}(k)) \to \mathbf{Ab} \qquad C_* \mapsto \mathrm{Hom}_{D_{\mathbb{A}^1}(k)}(\tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_X V), C_*)$$

is represented by $\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V)$.

Proof. The proof of [MS20, Proposition 2.37] extends to this level of generality. ☐

Corollary 4.4. *Let* $f : \mathrm{Th}_X V \to \mathrm{Th}_Y W$ *be a morphism in* $\mathcal{H}(k)_*^{\mathrm{cell}}$. *Then there exists a canonical chain homotopy class of morphisms*

$$\tilde{C}_*^{\mathrm{cell}}(f) : \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V) \to \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W)$$

in $\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))}(\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V), \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W))$.

Proof. Since $\tilde{C}_*^{\mathbb{A}^1}(-) : \mathcal{H}_*(k) \to D_{\mathbb{A}^1}(k)$ is a functor (see [Mor12, p. 161]), the morphism f induces a map $\tilde{C}_*^{\mathbb{A}^1}(f) : \tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_X V) \to \tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_Y W)$. Thus by Proposition 4.3, there is a map $\tilde{C}_*^{\mathrm{cell}}(f) : C_*^{\mathrm{cell}}(\mathrm{Th}_X V) \to \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W)$ in $D(\mathbf{Ab}_{\mathbb{A}^1}(k))$ such that the diagram

$$\begin{array}{ccc} \tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_X V) & \xrightarrow{\tilde{C}_*^{\mathbb{A}^1}(f)} & \tilde{C}_*^{\mathbb{A}^1}(\mathrm{Th}_Y W) \\ \phi \downarrow & & \phi \downarrow \\ \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V) & \xrightarrow[\tilde{C}_*^{\mathrm{cell}}(f)]{} & \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W) \end{array}$$

commutes. As noted previously, $\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V)$ is a bounded complex of projective objects of $\mathbf{Ab}_{\mathbb{A}^1}(k)$. It follows that $\mathrm{Hom}_{D(\mathbf{Ab}_{\mathbb{A}^1}(k))}(\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V), \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W))$ is the group of chain homotopy classes of maps in $\mathrm{Hom}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))}(\tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_X V), \tilde{C}_*^{\mathrm{cell}}(\mathrm{Th}_Y W))$. ☐

The following is the generalization of [MS20, Lemma 2.31] to include Thom spaces. The proof was omitted in [MS20], so we include one for completeness.

Lemma 4.5 (Künneth Formula)**.** *Suppose that* X *and* Y *are smooth schemes equipped with cellular structures, and let* p_1 *and* p_2 *be the projections of* $X \times_k Y$ *to* X *and* Y*, respectively. Let* V *and* W *be vector bundles on* X *and* Y*, respectively. Then there exists a cellular structure on* $X \times Y$ *and a natural isomorphism in* $D(\mathbf{Ab}^{p}_{\mathbb{A}^1}(k))$

$$\tilde{C}^{cell}_{*}(\mathrm{Th}_X V) \otimes_{D(\mathbf{Ab}^{p}_{\mathbb{A}^1}(k))} \tilde{C}^{cell}_{*}(\mathrm{Th}_Y W) \to \tilde{C}^{cell}_{*}(\mathrm{Th}_{X\times Y}(p_1^* V + p_2^* W)). \tag{8}$$

Proof. Let r and s be the (nonnegative) ranks of V and W, respectively. Let ν_i and μ_i denote the normal bundles of $X_i = \Omega_i(X) \setminus \Omega_{i-1}(X) \hookrightarrow \Omega_i(X)$ and $Y_i = \Omega_i(Y) \setminus \Omega_{i-1}(Y) \hookrightarrow \Omega_i(Y)$, respectively.

Note that there is a natural shifted cellular structure on $X \times Y$ equipped with $p_1^*V + p_2^*W \in K_0(X \times Y)$, where the open strata are $\Omega_n(X \times Y) = \cup_{i+j=n}(\Omega_i(X) \times \Omega_j(Y))$ and the closed strata are $(X \times Y)_n := \Omega_n(X \times Y) \setminus \Omega_{n-1}(X \times Y) = \sqcup_{i+j=n} X_i \times Y_j$. The normal bundle ξ_n on the inclusion $(X \times Y)_n \hookrightarrow \Omega_n(X \times Y)$ is the disjoint union of the normal bundle on each $X_i \times Y_j$, namely $\xi_n = \sqcup_{i+j=n}(p_1^*\nu_i + p_2^*\mu_j)$.

We compare the degree $n+r+s$ terms of the two sides of (8). The degree $n+r+s$ term of the right hand side of (8) is

$$\begin{aligned}
&\widetilde{H}^{\mathbb{A}^1}_{n+r+s}(\mathrm{Th}_{(X\times Y)_n}(p_1^*V + p_2^*W + \xi_n)) \\
&\qquad \cong \widetilde{H}^{\mathbb{A}^1}_{n+r+s}(\mathrm{Th}_{\sqcup_{i+j=n} X_i \times Y_j}(p_1^*V + p_2^*W + \sqcup_{i+j=n}(p_1^*\nu_i + p_2^*\mu_j))) \\
&\qquad \cong \widetilde{H}^{\mathbb{A}^1}_{n+r+s}(\vee_{i+j=n} \mathrm{Th}_{X_i\times Y_j}(p_1^*V + p_2^*W + p_1^*\nu_i + p_2^*\mu_j)) \\
&\qquad \cong \bigoplus_{i+j=n} \widetilde{H}^{\mathbb{A}^1}_{n+r+s}(\mathrm{Th}_{X_i\times Y_j}(p_1^*V + p_2^*W + p_1^*\nu_i + p_2^*\mu_j)).
\end{aligned}$$

The degree $n+r+s$ term of the left hand side of (8) is

$$\bigoplus_{i+j=n+r+s} \tilde{C}^{cell}_{i}(\mathrm{Th}_X V) \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} \tilde{C}^{cell}_{j}(\mathrm{Th}_Y W) \cong \bigoplus_{i+j=n} \widetilde{H}^{\mathbb{A}^1}_{i+r}(\mathrm{Th}_{X_i}(V+\nu_i)) \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} \widetilde{H}^{\mathbb{A}^1}_{j+s}(\mathrm{Th}_{Y_j}(W+\mu_j)).$$

For any X' and Y' with $V' \in K_0(X')$ and $W' \in K_0(Y')$, we have a natural equivalence $\mathrm{Th}_{X'}(V') \times \mathrm{Th}_{Y'}(W') \to \mathrm{Th}_{X'\times Y'}(V' \times W')$. We therefore have an induced map

$$\begin{aligned}
&\widetilde{H}^{\mathbb{A}^1}_{i+r}(\mathrm{Th}_{X_i}(V + \nu_i)) \otimes_{\mathbf{Ab}_{\mathbb{A}^1}(k)} \widetilde{H}^{\mathbb{A}^1}_{j+s}(\mathrm{Th}_{Y_j}(W + \mu_j)) \\
&\qquad \to \widetilde{H}^{\mathbb{A}^1}_{i+j+r+s}(\mathrm{Th}_{X_i\times Y_j}(p_1^*V + p_2^*W + p_1^*\nu_i + p_2^*\mu_j)).
\end{aligned} \tag{9}$$

Unwinding the definitions and using the functoriality of the Thom space, (9) induces a map of chain complexes (8), i.e. is compatible with the differentials.

We claim that for $i+j=n$ the map (9) is an isomorphism. By [MS20, Lemma 2.13], both $V+\nu_i$ and $W+\mu_j$ represent trivial elements of $K_0(X_i)$ and $K_0(Y_j)$ respectively. Choosing trivializations induces a trivialization of $p_1^*V + p_2^*W + p_1^*\nu_i + p_2^*\mu_j$ on $X_i \times Y_j$. These trivializations and the suspension isomorphism for $\mathbb{A}^1$-homology [Mor12, Remark 6.30] induce the following isomorphisms:

$$\widetilde{H}^{\mathbb{A}^1}_{i+r}(\mathrm{Th}_{X_i}(V+\nu_i)) \cong \widetilde{H}^{\mathbb{A}^1}_{i+r}((X_i)_+ \wedge (\mathbb{G}_m)^{\wedge(i+r)} \wedge (S^1)^{\wedge(i+r)}) \cong \widetilde{H}^{\mathbb{A}^1}_{0}((X_i)_+ \wedge (\mathbb{G}_m)^{\wedge(i+r)})$$

$$\widetilde{H}^{\mathbb{A}^1}_{j+s}(\mathrm{Th}_{Y_j}(W+\mu_j)) \cong \widetilde{H}^{\mathbb{A}^1}_{j+s}((Y_j)_+ \wedge (\mathbb{G}_m)^{\wedge(j+s)} \wedge (S^1)^{\wedge(j+s)}) \cong \widetilde{H}^{\mathbb{A}^1}_{0}((Y_j)_+ \wedge (\mathbb{G}_m)^{\wedge(j+s)})$$

$$
\begin{aligned}
\widetilde{H}^{\mathbb{A}^1}_{i+j+r+s}(\mathrm{Th}_{X_i\times Y_j}(p_1^*V + p_2^*W + p_1^*\nu_i + p_2^*\mu_j))&\\
&\cong \widetilde{H}^{\mathbb{A}^1}_{i+j+r+s}((X_i\times Y_j)_+\wedge(\mathbb{G}_m)^{\wedge(i+j+r+s)}\wedge(S^1)^{\wedge(i+j+r+s)})\\
&\cong \widetilde{H}^{\mathbb{A}^1}_0((X_i\times Y_j)_+\wedge(\mathbb{G}_m)^{\wedge(i+j+r+s)}).
\end{aligned}
$$

Note that $\widetilde{H}^{\mathbb{A}^1}_0$ takes a sheaf of sets to the free strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on the sheaf of sets, and thus transforms $\wedge$ to $\otimes$. The claim follows. The differentials are as claimed by the usual Puppe sequence argument. □

Lemma 4.6. *Suppose that X and Y are smooth schemes equipped with cellular structures, and let p_1 and p_2 be the projections of $X\times_k Y$ to X and Y, respectively. Let V and W be vector bundles on X and Y, respectively. Then $\tilde{C}^{\mathrm{cell}}_*$ respects the symmetry maps $\tau_{\mathcal{H}^{\mathrm{cell}}_*(k)}(\mathrm{Th}_X V, \mathrm{Th}_Y W)$ and $\tau_{D(\mathbf{Ab}^p_{\mathbb{A}^1}(k))}$ in the sense that the diagram*

$$
\begin{array}{ccc}
\tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_X V)\otimes\tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_Y W) & \xrightarrow[m_1]{\cong} & \tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_{X\times Y}\, p_1^*V + p_2^*W)\\
\downarrow{\scriptstyle \tau_{D(\mathbf{Ab}^p_{\mathbb{A}^1}(k))}} & & \downarrow{\scriptstyle \tilde{C}^{\mathrm{cell}}_*(\tau_{\mathcal{H}^{\mathrm{cell}}_*})}\\
\tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_Y W)\otimes\tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_X V) & \xrightarrow[m_2]{\cong} & \tilde{C}^{\mathrm{cell}}_*(\mathrm{Th}_{Y\times X}\, p_2^*W + p_1^*V)
\end{array}
\tag{10}
$$

commutes.

Proof. With appropriate orientation data, (10) becomes a sum of diagrams of the form

$$
\begin{array}{ccc}
H^{\mathbb{A}^1}_0(X_{i-r})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_i\otimes_{\mathbb{A}^1}H^{\mathbb{A}^1}_0(Y_{j-s})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_j & \longrightarrow & H^{\mathbb{A}^1}_0(X_{i-r}\times Y_{j-s})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_n\\
\downarrow{\scriptstyle \tau_{D(\mathbf{Ab}^p_{\mathbb{A}^1}(k))}} & & \downarrow{\scriptstyle \tilde{C}^{\mathrm{cell}}_*(\tau_{\mathcal{H}^{\mathrm{cell}}_*})}\\
H^{\mathbb{A}^1}_0(Y_{j-s})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_j\otimes_{\mathbb{A}^1}H^{\mathbb{A}^1}_0(X_{i-r})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_i & \longrightarrow & H^{\mathbb{A}^1}_0(Y_{j-s}\times X_{i-r})\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_n
\end{array}
\tag{11}
$$

where $i+j=n$, $i,j\geq 1$, along with similar diagrams where i or j is 0 and the appropriate factors of $\underline{K}^{\mathrm{MW}}_*$ are omitted. In (11) the map $\tilde{C}^{\mathrm{cell}}_*(\tau_{\mathcal{H}^{\mathrm{cell}}_*})$ is induced by applying $\widetilde{H}^{\mathbb{A}^1}_n$ to a swap map on spaces of the form $(X_i)_+\wedge S^{2i,i}\wedge(Y_j)_+\wedge S^{2j,j}$ with X_i and Y_j cohomologically trivial. This map is multiplication by the swap map $S^{2i,i}\wedge S^{2j,j}\to S^{2j,j}\wedge S^{2i,i}$ in $\mathrm{GW}(k)$. This element of $\mathrm{GW}(k)$ equals $\langle -1\rangle^{ij}$ by [Mor12, Lemma 3.43(2)], which states that $\tau(S^{p_1,q_1},S^{p_2,q_2})$ represents $(-1)^{(p_1-q_1)(p_2-q_2)}(-\langle -1\rangle)^{q_1q_2}$ in $\mathrm{GW}(k)$. By definition, the map $\tau_{D(\mathbf{Ab}^p_{\mathbb{A}^1}(k))}$ is multiplication by $(-1)^{ij}$ times the canonical swap on a tensor product (see (7)).

The map m_1 is the tensor product of the isomorphism $H^{\mathbb{A}^1}_0(X_{i-r})\otimes_{\mathbb{A}^1}H^{\mathbb{A}^1}_0(Y_{j-s})\to H^{\mathbb{A}^1}_0(X_{i-r}\times Y_{j-s})$ with the isomorphism $\underline{K}^{\mathrm{MW}}_i\otimes_{\mathbb{A}^1}\underline{K}^{\mathrm{MW}}_j\to\underline{K}^{\mathrm{MW}}_n$ induced by multiplication on Milnor–Witt K-theory. A similar statement holds for m_2 where the factors are reversed. Since Milnor–Witt K-theory is $-\langle -1\rangle$ graded commutative [Mor12, Corollary 3.8], it follows that m_2 is a canonical swap on a tensor product to reorder the factors followed by $(-\langle -1\rangle)^{ij}$ times m_1. Since $(-1)^{ij}(-\langle -1\rangle)^{ij}=\langle -1\rangle^{ij}$, the claim follows. □

Proposition 4.7. *The functor $\tilde{C}^{\mathrm{cell}}_*:\mathcal{H}(k)^{\mathrm{cell}}_*\to D(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$ is symmetric monoidal.*

Proof. This follows from Corollary 4.4, Lemma 4.5, and Lemma 4.6. □

5. Spanier–Whitehead cellular complex

5.1. **Definitions and basic properties.** Recall we defined $\mathrm{SW}^{\mathrm{cell}}(k) := \mathcal{H}(k)^{\mathrm{cell}}_*[(\mathbb{P}^1)^{\otimes -1}]$ in Section 3.2. Since $\tilde{C}^{\mathrm{cell}}_*$ is a symmetric monoidal functor and the cyclic permulation of $\mathbb{P}^1 \otimes \mathbb{P}^1 \otimes \mathbb{P}^1$ is the identity in $\mathcal{H}(k)_*$, the cyclic permutation of $\tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1) \otimes \tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1) \otimes \tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1)$ is the identity in $\mathrm{D}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$. Denote the corresponding Spanier–Whitehead category (as in Definition 2.5) by

$$\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k)) := \mathrm{D}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))[\tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1)^{\otimes -1}].$$

By [MS20, Corollary 2.51], the complex $C^{\mathrm{cell}}_*(\mathbb{P}^1) \simeq \tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1_+)$ is represented by the complex

$$\underline{K}^{\mathrm{MW}}_1 \xrightarrow{0} \mathbb{Z}.$$

Thus $\tilde{C}^{\mathrm{cell}}_*(\mathbb{P}^1) \simeq \underline{K}^{MW}_1[1]$ and we have

$$\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k)) \simeq \mathrm{D}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))[(\underline{K}^{MW}_1[1])^{\otimes -1}]. \tag{12}$$

Definition 5.1. Define

$$C^{\text{SW-cell}} : \mathrm{SW}^{\mathrm{cell}}(k) \to \mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$$

to be the symmetric monoidal functor obtained by applying Proposition 2.6 to the symmetric monoidal functor $\tilde{C}^{\mathrm{cell}}_* : \mathcal{H}(k)^{\mathrm{cell}}_* \to \mathrm{D}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$ of Proposition 4.7.

Corollary 5.2. *Let X be smooth and projective over a field k and suppose that X admits a cellular structure. For any endomorphism $\varphi\colon X \to X$ and any integer $m \geq 1$, we have*

$$\mathrm{Tr}(C^{\text{SW-cell}}(\varphi^m)) = C^{\text{SW-cell}}(\mathrm{Tr}(\varphi^m)).$$

Proof. We may assume $m = 1$. By Proposition 3.6, X is dualizable in $\mathrm{SW}^{\mathrm{cell}}(k)$. The result follows by applying Proposition 2.4 to the symmetric monoidal functor $C^{\text{SW-cell}}$. □

Proposition 5.3. *The morphism*

$$C^{\text{SW-cell}} : \mathrm{End}(1_{\mathrm{SW}^{\mathrm{cell}}(k)}) \to \mathrm{End}(1_{\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))}) \tag{13}$$

is an isomorphism, and both sides are isomorphic to $\mathrm{GW}(k)$.

Proof. We compute $C^{\text{SW-cell}}(1_{\mathrm{SW}^{\mathrm{cell}}(k)}) = C^{\text{SW-cell}}(\mathrm{Spec}\, k_+) = \mathbb{Z}$. By the identification (12), we obtain

$$\begin{aligned}
&\mathrm{End}(1_{\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))}) \\
&\quad= \operatorname*{colim}_{n\geq 0} \mathrm{End}_{\mathrm{D}(\mathbf{Ab}_{\mathbb{A}^1}(k))}(\underline{K}^{MW}_1[1]^{\otimes n}) \\
&\quad\cong \operatorname*{colim}_{n\geq 0} \mathrm{End}_{\mathrm{D}(\mathbf{Ab}_{\mathbb{A}^1}(k))}(\underline{K}^{MW}_n[n]) && \text{(by [Mor12, Theorem 3.37])} \\
&\quad\cong \operatorname*{colim}_{n\geq 0} \mathrm{End}_{\mathbf{Ab}_{\mathbb{A}^1}(k)}(\underline{K}^{MW}_n) && \text{(since } \underline{K}^{MW}_n \text{ is projective by [Mor12, Theorem 3.37])} \\
&\quad\cong \operatorname*{colim}_{n\geq 0} \mathrm{GW}(k) && \text{(by Lemma 2.7)} \\
&\quad\cong \mathrm{GW}(k).
\end{aligned}$$

We have $\mathrm{End}(1_{\mathrm{SW}^{\mathrm{cell}}(k)}) = \mathrm{colim}_{n\geq 0}\, \mathrm{End}(S^{2n,n}) \cong \mathrm{GW}(k)$ by [Mor12, Cor 6.43]. Since $H^{\mathbb{A}^1}_*$ respects the $\mathrm{GW}(k)$-module structure on $\mathrm{End}(S^{2n,n})$, so does $C^{\text{SW-cell}}$. Thus the map (13) becomes the identity map on $\mathrm{GW}(k)$ under the given isomorphisms. □

5.2. **Endomorphisms, traces, and characteristic polynomials.** Given an endomorphism φ of a dualizable object in $\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$, we can use the categorial trace of φ and its powers to define a logarithmic characteristic polynomial of φ, which, using Proposition 5.3, is a power series with coefficients in $\mathrm{GW}(k)$.

Definition 5.4. The logarithmic characteristic polynomial of an endomorphism φ of a dualizable object in $\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$ is the power series defined by

$$\operatorname{dlog} P_{\varphi}(t) := \sum_{m=1}^{\infty} -\operatorname{Tr}(\varphi^m)t^{m-1} \in \mathrm{GW}(k)[[t]].$$

We use the convention that dlog indicates a formal logarithmic derivative, while $\frac{d}{dt}\log$ denotes the derivative of the logarithm.

This definition is motivated by the usual definition of the characteristic polynomial of a square matrix.

Definition 5.5. For any commutative ring R and endomorphism φ of R^n represented by a matrix A, the characteristic polynomial $P_{\varphi}(t)$ is defined as $P_{\varphi}(t) := \det(1 - At)$.

We then have the following elementary relation.

Lemma 5.6. *Let R be a commutative ring and let $A \colon R^n \to R^n$ be an endomorphism. Then*

$$\frac{d}{dt}\log(P_A(t)) = \sum_{m=1}^{\infty} -\operatorname{Tr}(A^m)t^{m-1}.$$

Proof. We will show that these two power series agree by comparing each coefficient. The coefficient of t^i on each side is a polynomial (with integer coefficients) in the entries of the matrix representing the endomorphism A. To show that these integer polynomials agree, it suffices to prove that they take the same value on every complex number. In other words, it suffices to prove the theorem when $R = \mathbb{C}$. In this case, we may assume that the matrix of A is upper-triangular with diagonal entries $c_1, \ldots, c_n$ (since trace is independent of a choice of basis). We have $P_A(t) = \prod_{i=1}^n (1 - tc_i)$ and $\operatorname{Tr}(A^m) = \sum_{i=1}^n c_i^m$, which gives

$$\begin{aligned}\frac{d}{dt}\log(P_A(t)) &= \sum_{i=1}^{n}\left(-\sum_{m=1}^{\infty} c_i^m t^{m-1}\right)\\ &= \sum_{m=1}^{\infty}\left(-\sum_{i=1}^{n} c_i^m\right)t^{m-1}\\ &= \sum_{m=1}^{\infty} -\operatorname{Tr}(A^m)t^{m-1}.\end{aligned}$$

□

Let C_* in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))$ be a bounded complex such that $C_n \cong (\underline{K}_n^{MW})^{b_n}$ for some nonnegative integers b_n. For example, we may take $C_* = C_*^{\mathrm{cell}}(X)$ for a smooth simple cellular scheme X (Remark 4.1). Let N be an integer. Let t be a nonnegative integer such that $C_n = 0$ for $n \geq t$. Define

$$\mathbb{D}(C_*, N) := [\underline{\mathrm{Hom}}_{\mathbf{Ch}_{\geq 0}(\mathbf{Ab}(k))}(C_*, \underline{K}_t^{MW}[t]), -t - N].$$

By Lemma 2.7, $\mathbb{D}(C_*, N)$ is a representative for an object in $\mathrm{D}^{\mathrm{SW}}(\mathbf{Ab}^{\mathrm{p}}_{\mathbb{A}^1}(k))$.

Proposition 5.7. *The object* $\mathbb{D}(C_*, N)$ *is a dual object to* (C_*, N) *in* $D^{SW}(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$ *in the sense of Definition 2.1.*

Proof. Define $\epsilon\colon \mathbb{D}(C_*, N) \otimes (C_*, N) \to (\mathbb{Z}, 0)$ to be the map in $D^{SW}(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$ associated to the natural evaluation map

$$\epsilon[t] : \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_*, \underline{K}^{MW}_t[t]) \otimes C_* \to \underline{K}^{MW}_t[t] \cong \underline{K}^{MW}_1[1]^{\otimes t}$$

in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}(k))$.

Define a map $\underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n) \otimes \underline{K}^{MW}_t \to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t)$ by $\phi \otimes a \mapsto \phi_a$ where $\phi_a(c) = \phi(c) \otimes a$. Since $C_n \cong (\underline{K}^{MW}_n)^{b_n}$, $\underline{K}^{MW}_n \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t \cong \underline{K}^{MW}_{n+t}$ [Mor12, Theorem 3.37], and $\underline{\mathrm{Hom}}(\underline{K}^{MW}_n, \underline{K}^{MW}_{n+t}) \cong \underline{K}^{MW}_t$ by Lemma 2.7, we find that $\underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t)$ is strictly $\mathbb{A}^1$-invariant. (Note that $t - n \geq 1$ by construction.) This defines a map

$$f_n : \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n) \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t \to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t)$$

By similar reasoning, we have a map

$$g_n : \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) \otimes_{\mathbb{A}^1} C_n \to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t \otimes_{\mathbb{A}^1} C_n)$$

Since $t > n$, the map g is an isomorphism. (This can be seen using the isomorphisms $C_n \cong (\underline{K}^{MW}_n)^{b_n}$, $\underline{\mathrm{Hom}}(\underline{K}^{MW}_n, \underline{K}^{MW}_t) \cong \underline{K}^{MW}_{t-n}$, and $\underline{K}^{MW}_{t-n} \otimes_{\mathbb{A}^1} \underline{K}^{MW}_n \cong \underline{K}^{MW}_t$, etc.) Let

$$h_n : \underline{K}^{MW}_t \to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n) \otimes \underline{K}^{MW}_t$$

denote the map defined by $a \mapsto 1_{C_n} \otimes a$.

The composite $\tau(n, t-n) \circ g_n^{-1} \circ \tau(n, t) \circ f_n \circ h_n$ defines a map

$$\begin{aligned} \underline{K}^{MW}_t &\to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n) \otimes \underline{K}^{MW}_t \\ &\to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, C_n \otimes_{\mathbb{A}^1} \underline{K}^{MW}_t) \xrightarrow{(-\langle -1\rangle)^{nt}} \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t \otimes_{\mathbb{A}^1} C_n) \\ &\to \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) \otimes_{\mathbb{A}^1} C_n \xrightarrow{(-\langle -1\rangle)^{n(t-n)}} C_n \otimes_{\mathbb{A}^1} \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) \end{aligned}$$

where $\tau(i, j)$ is the swap $a \otimes b \mapsto (-\langle -1\rangle)^{ij} b \otimes a$. The sign comes from the graded $(-\langle -1\rangle)$-commutativity of $\underline{K}^{MW}_*$. See [Mor12, Corollary 3.8].

Taking the product of the $\tau(n, t-n) \circ g_n^{-1} \circ \tau(n, t) \circ f_n \circ h_n$ over n defines a map

$$\eta[t] : \underline{K}^{MW}_t[t] \to C_* \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} \underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t]).$$

Let $\eta\colon (\mathbb{Z}, 0) \to (C_*, N) \otimes \mathbb{D}(C_*, N)$ be the associated map in $D^{SW}(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$. We leave checking that these maps satisfy the desired properties as in Definition 2.1 to the reader. ☐

Let C_* in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))$ be a bounded complex such that $C_i \cong (\underline{K}^{MW}_i)^{b_i}$ for some nonnegative integers b_i. Let $\varphi : C_* \to C_*$ be a morphism in $D(\mathbf{Ab}_{\mathbb{A}^1}(k))$. As in the proof of Corollary 4.4, the morphism φ is represented by a map of complexes, so we have maps $\varphi_i : (K^{MW}_i)^{b_i} \to (K^{MW}_i)^{b_i}$. Such a φ_i is determined by an $b_i \times b_i$ square matrix of elements of $\mathrm{Hom}(K^{MW}_i, K^{MW}_i) \cong GW(k)$ (Lemma 2.7). The trace of such a matrix is denoted Tr and defined to be the sum of the diagonal entries.

Proposition 5.8. *Let* C_* *in* $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))$ *be a bounded complex such that* $C_n \cong (\underline{K}^{MW}_n)^{b_n}$ *for some nonnegative integers* b_n*. Let* $\varphi : C_* \to C_*$ *be a morphism in* $D(\mathbf{Ab}_{\mathbb{A}^1}(k))$*. The* *categorical trace of the corresponding map* φ *in* $D^{SW}(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$ *is computed as*

$$\mathrm{Tr}(\varphi) = \sum_{i=-\infty}^{\infty} \langle -1 \rangle^i \,\mathrm{Tr}(\varphi_i).$$

Proof. Let ϵ, η be the evaluation and coevaluation maps as in Proposition 5.7, and τ the symmetry isomorphism in the category $D^{SW}(\mathbf{Ab}^p_{\mathbb{A}^1}(k))$ as in (7). Then $\mathrm{Tr}(\varphi)$ is the composition

$$1 \xrightarrow{\eta} (C_*, N) \otimes \mathbb{D}(C_*, N) \xrightarrow{\varphi\otimes 1} (C_*, N) \otimes \mathbb{D}(C_*, N) \xrightarrow{\tau} \mathbb{D}(C_*, N) \otimes (C_*, N) \xrightarrow{\epsilon} 1.$$

This map is represented by a map

$$\begin{aligned}\underline{K}^{MW}_t[t] \to C_* \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} \underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t]) &\xrightarrow{\varphi\otimes 1} C_* \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} \underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t]) \\ &\to \underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t]) \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} C_* \to \underline{K}^{MW}_t[t]\end{aligned}$$

in $\mathbf{Ch}_{\geq 0}(\mathbf{Ab}_{\mathbb{A}^1}(k))$, where t is as in Proposition 5.7 and its proof. This map is concentrated in degree t. The degree t sheaves of the complexes $C_* \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} \underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t])$ and $\underline{\mathrm{Hom}}(C_*, \underline{K}^{MW}_t[t]) \otimes_{\mathbf{Ch}(\mathbf{Ab}_{\mathbb{A}^1}(k))} C_*$ are isomorphic to a direct sum over n of $C_n \otimes_{\mathbb{A}^1} \underline{\mathrm{Hom}}(C_n, \underline{K}^{MW}_t)$, leading to an expression for $\mathrm{Tr}(\varphi)$ as the sum over n of maps

$$\begin{aligned}\underline{K}^{MW}_t \to C_n \otimes_{\mathbb{A}^1} \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) &\xrightarrow{\varphi\otimes 1} C_n \otimes_{\mathbb{A}^1} \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) \\ &\xrightarrow{(-1)^n} \underline{\mathrm{Hom}}_{\mathbf{Ab}(k)}(C_n, \underline{K}^{MW}_t) \otimes C_n \to \underline{K}^{MW}_t.\end{aligned} \tag{14}$$

Tracing through the definitions of Proposition 5.7, the composite (14) is

$$(-\langle -1 \rangle)^{tn} (-\langle -1 \rangle)^{(t-n)n} (-1)^n \,\mathrm{Tr}(\varphi_n) = \langle -1 \rangle^n \,\mathrm{Tr}(\varphi_n).$$

Thus $\mathrm{Tr}(\varphi) = \sum_{n=-\infty}^{\infty} \langle -1 \rangle^i \,\mathrm{Tr}(\varphi_n)$ as claimed. □

5.3. **Cellular Grothendieck–Lefschetz trace formula.** Hoyois [Hoy15] proves a quadratic refinement of the Grothendieck–Lefschetz trace formula in the setting of stable motivic homotopy theory, in the sense of relating the trace of an endomorphism $\varphi : X \to X$ of a smooth proper scheme to the fixed points of φ. The machinery of Morel–Sawant and the above give an expression for the trace in terms of traces of matrices of elements of $GW(k)$ for simple cellular X:

Theorem 5.9. *Let* X *be a smooth projective scheme over a field* k *and suppose that* X *admits a simple cellular structure. Let* $\varphi \colon X \to X$ *be an endomorphism, and let* $C^{cell}_*(\varphi) : C^{cell}_*(X) \to C^{cell}_*(X)$ *be any representative of the canonical chain homotopy class. We have the equality*

$$\mathrm{Tr}(\varphi) = \sum_i \langle -1 \rangle^i \,\mathrm{Tr}(C^{cell}_i(\varphi))$$

in $GW(k)$.

Note that with our notational conventions, $C^{cell}_*(X) \simeq \tilde{C}^{cell}_*(X_+) \simeq C^{SW\text{-}cell}_*(X)$.

Proof. By Lemma 3.6, the scheme X is fully dualizable and $\mathrm{Tr}(\varphi)$ is a well-defined element of $GW(k)$. By Proposition 5.3, there is an equality $\mathrm{Tr}(\varphi) = C^{SW\text{-}cell}_*(\mathrm{Tr}(\varphi))$. By Corollary 5.2, $C^{SW\text{-}cell}_*(\mathrm{Tr}(\varphi)) = \mathrm{Tr}(C^{SW\text{-}cell}_*(\varphi))$. By Remark 4.1, $C^{SW\text{-}cell}_*(X) \simeq C^{cell}_*(X)$ satisfies the hypotheses of Proposition 5.8. Applying Proposition 5.8 proves the theorem. □

Classical Lefschetz trace formulas in algebraic or topological categories take the form

$$\sum_i (-1)^i \operatorname{Tr}(\varphi^*|_{H^i(X)}) = \operatorname{Tr}(\varphi) = \sum_{x:\varphi(x)=x} \operatorname{ind}_x \varphi$$

under appropriate hypotheses. For simple cellular schemes, Theorem 5.9 gives a quadratic refinement of the left equality. We will combine this with Hoyois's enrichment of the right equality later in Section 8.2.

6. Rationality of the zeta function

In this section, we prove that the logarithmic zeta function of an endomorphism of a scheme with a cellular structure is computed via $C_*^{\text{SW-cell}}(X)$ (Theorem 6.1). When the cellular structure is moreover simple, we more explicitly show that the logarithmic zeta function is computed by the action on the terms $C_i^{\text{cell}}(X)$ of Morel–Sawant's cellular complex (Theorem 6.2). This, in turn, proves that $\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1}$ is dlog rational (cf. Definition 1.2) in this case.

Let X be a cellular scheme over a field k (cf. Definition 3.5) with an endomorphism $\varphi\colon X \to X$. Since $C_*^{\text{SW-cell}}$ is a functor, we obtain an endomorphism

$$C_*^{\text{SW-cell}}(\varphi)\colon C_*^{\text{SW-cell}}(X) \to C_*^{\text{SW-cell}}(X).$$

Using Definition 5.4, we define

$$\operatorname{dlog} P_{X,\varphi}^{\text{SW-cell}}(t) = \operatorname{dlog} P_{C_*^{\text{SW-cell}}(\varphi)}(t) = \sum_{m=1}^{\infty} -\operatorname{Tr}(C_*^{\text{SW-cell}}(\varphi)^m)t^{m-1}$$

to be the logarithmic characteristic polynomial of the endomorphism $C_*^{\text{SW-cell}}(\varphi)$.

Theorem 6.1. *Let X be a smooth projective cellular scheme over a field k and let $\varphi\colon X \to X$ be an endomorphism. Then*

$$\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1}(t) = -\operatorname{dlog} P_{X,\varphi}^{\text{SW-cell}}(t).$$

Proof. We compute

$$\begin{aligned}
\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1}(t) &= \sum_{m\geq 1} \operatorname{Tr}(\varphi^m)t^{m-1} && \text{(by definition)}\\
&= \sum_{m\geq 1} C_*^{\text{SW-cell}}(\operatorname{Tr}(\varphi^m))t^{m-1} && \text{(by proof of Proposition 5.3)}\\
&= \sum_{m\geq 1} \operatorname{Tr}(C_*^{\text{SW-cell}}(\varphi^m))t^{m-1} && \text{(by Corollary 5.2)}\\
&= \sum_{m\geq 1} \operatorname{Tr}(C_*^{\text{SW-cell}}(\varphi)^m)t^{m-1} && \text{(by functoriality)}\\
&= -\operatorname{dlog} P_{X,\varphi}^{\text{SW-cell}}(t) && \text{(by definition).}
\end{aligned}$$ □

We can make $\operatorname{dlog} P_{X,\varphi}^{\text{SW-cell}}(t)$ more explicit when the cellular structure is simple. Assume this is the case. Let

$$0 \subsetneq \Omega_0(X) \subsetneq \cdots \subsetneq \Omega_s(X) = X$$

be a simple cellular structure on X. Write $X_i := \Omega_i(X) \smallsetminus \Omega_{i-1}(X)$. Then

$$C_i^{\text{SW-cell}}(X) := (C_i^{\text{cell}}(X), 0) \qquad \text{and} \qquad C_i^{\text{cell}}(X) \cong \begin{cases} (\underline{K}_i^{\text{MW}})^{b_i} & i > 0 \\ \mathbb{Z}^{b_0} & i = 0. \end{cases}$$

See Remark 4.1. By Corollary 4.4 or [MS20, Corollary 2.43], there is a canonical chain homotopy class of endomorphisms $C_*^{\text{cell}}(\varphi)$ of $C_*^{\text{cell}}(X)$. Choosing a representative, we obtain an endomorphism

$$C_i^{\text{cell}}(\varphi)\colon C_i^{\text{cell}}(X) \to C_i^{\text{cell}}(X)$$

of each term in the complex $C_*^{\text{cell}}(X)$. Since $C_i^{\text{cell}}(X) \simeq (\underline{K}_i^{MW})^{b_i}$, the endomorphism $C_i^{\text{cell}}(\varphi)$ is determined by a $b_i \times b_i$ square matrix of entries in $\mathrm{Hom}(\underline{K}_i^{MW}, \underline{K}_i^{MW}) \cong \mathrm{GW}(k)$, which is a commutative ring. We therefore have the usual notion of the characteristic polynomial $P_{C_i^{\text{cell}}(\varphi)}(t)$ of $C_i^{\text{cell}}(\varphi)$ in the polynomial ring over $\mathrm{GW}(k)$, namely

$$P_{C_i^{\text{cell}}(\varphi)}(t) = \det(1 - tC_i^{\text{cell}}(\varphi)).$$

Theorem 6.2. *Let X be a smooth projective simple cellular scheme over a field k and let $\varphi : X \to X$ be an endomorphism. Then*

$$\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1} = \sum_{i=-\infty}^{\infty} -\langle -1 \rangle^i \frac{d}{dt} \log P_{C_i^{\text{cell}}(\varphi)}(t).$$

Proof. We have

$$\begin{aligned}
\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1}(t) &= \sum_{m \geq 1} \mathrm{Tr}(C_*^{\text{SW-cell}}(\varphi^m))t^{m-1} && \text{(by Theorem 6.1)} \\
&= \sum_{m \geq 1} \sum_{i=-\infty}^{\infty} \langle -1 \rangle^i \, \mathrm{Tr}(C_i^{\text{SW-cell}}(\varphi^m))t^{m-1} && \text{(by Proposition 5.8)} \\
&= \sum_{m \geq 1} \sum_{i=-\infty}^{\infty} \langle -1 \rangle^i \, \mathrm{Tr}(C_i^{\text{cell}}(\varphi^m))t^{m-1} && \text{(by construction)} \\
&= \sum_{m \geq 1} \sum_{i=-\infty}^{\infty} \langle -1 \rangle^i \, \mathrm{Tr}(C_i^{\text{cell}}(\varphi)^m)t^{m-1} && \text{(by Corollary 4.4)} \\
&= \sum_{i=-\infty}^{\infty} -\langle -1 \rangle^i \frac{d}{dt} \log P_{C_i^{\text{cell}}(\varphi)}(t) && \text{(by Lemma 5.6).}
\end{aligned}$$

□

Remark 6.3. If a power series $\Phi(t)$ is dlog rational in the sense of Definition 1.2, it is tempting to write $\Phi = \operatorname{dlog} \prod_j P_j^{c_j}$, where the power operation should be a suitable power structure on the Grothendieck–Witt ring. In fact, the ring $\mathrm{GW}(k)$ is endowed with a natural pre-λ-ring structure

$$\lambda_t : \mathrm{GW}(k) \to 1 + t\,\mathrm{GW}(k)[[t]]$$

sending each generator $\langle a \rangle$ to $1 + \langle a \rangle t$ (it is induced by sending a quadratic space to its successive alternating powers). By [GZLMH06], this endows $\mathrm{GW}(k)$ with a power structure such that

$$\left(\frac{1}{1-t}\right)^r = \lambda_t(r)$$

for every $r \in \mathrm{GW}(k)$. In particular, we have

$$(1-t)^{-\langle a\rangle} = 1 + \langle a\rangle t.$$

Taking logarithmic derivatives, we get

$$\mathrm{dlog}\left((1-t)^{-\langle a\rangle}\right) = \frac{\langle a\rangle}{1+\langle a\rangle t}.$$

We conclude from this that this power structure is not compatible with logarithmic derivatives, in the sense that in general for $r \in \mathrm{GW}(k)$ and $f \in 1 + t\,\mathrm{GW}(k)[[t]]$ we do not have

$$\mathrm{dlog}\, f^r = r\,\mathrm{dlog}\, f.$$

Thus, this natural power structure with $\mathrm{GW}(k)$ does not provide us with a way of lifting our definition of dlog rationality to a notion of rationality.

7. $\mathbb{A}^1$-logarithmic zeta functions and real points

Let $r^{\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}$ denote the real realization functor from the stable $\mathbb{A}^1$-homotopy category over $\mathbb{R}$ to the topological stable homotopy category [Bac18, Section 10] . As above, we use the isomorphism $\mathrm{End}(1_{\mathrm{SH}(\mathbb{R})}) \cong \mathrm{GW}(\mathbb{R})$ given by the $\mathbb{A}^1$-degree of Morel. This $\mathbb{A}^1$-degree has the beautiful property (see, e.g., [AFW20, Prop 3.1.3]) that it encodes the topological degree of the corresponding map on real points:

$$\mathrm{sign}\,\mathrm{deg}^{\mathbb{A}^1}(f) = \mathrm{deg}^{\mathrm{top}}\, r^{\mathbb{R}}(f), \tag{15}$$

where $\mathrm{deg}^{\mathrm{top}}$ denotes the topological degree of a self map of a sphere, and $\mathrm{sign} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$ is the signature homomorphism, which takes the class of a bilinear form over $\mathbb{R}$ to its signature. So $\mathrm{sign}(a\langle 1\rangle + b\langle -1\rangle) = a - b$ for all $a, b \in \mathbb{Z}$. In other works, the map $r^{\mathbb{R}}(\mathrm{End}_1)$ induced by the real realization on endomorphisms of the corresponding unit objects $r^{\mathbb{R}}(\mathrm{End}_1)\colon\ \mathrm{GW}(\mathbb{R}) \cong \mathrm{End}(1_{\mathrm{SH}(\mathbb{R})}) \to \mathrm{End}(1_{\mathrm{SH}}) \cong \mathbb{Z}$ is $\mathrm{sign} : \mathrm{GW}(\mathbb{R}) \to \mathbb{Z}$.

Lemma 7.1. *Suppose that $\varphi : X \to X$ is an endomorphism of a smooth scheme X over $\mathbb{R}$. Let $X(\mathbb{R})$ denote the real points of X, viewed as a real manifold, and $\varphi(\mathbb{R}) : X(\mathbb{R}) \to X(\mathbb{R})$ the corresponding endomorphism. Then*

$$\mathrm{sign}\,\mathrm{Tr}(\varphi) = \mathrm{Tr}(\varphi(\mathbb{R})).$$

Proof. The real realization functor $r^{\mathbb{R}}$ is symmetric monoidal by [HO16, Section 4]. Thus $r^{\mathbb{R}}\,\mathrm{Tr}(\varphi) = \mathrm{Tr}(r^{\mathbb{R}}(\varphi)) = \mathrm{Tr}(\varphi(\mathbb{R}))$ by, e.g., Proposition 2.4. Then $r^{\mathbb{R}}\,\mathrm{Tr}(\varphi) = \mathrm{sign}\,\mathrm{Tr}(\varphi)$ by (15). □

Proposition 7.2. *Suppose that $\varphi : X \to X$ is an endomorphism of a smooth projective scheme X over $\mathbb{R}$. Then in $\mathbb{Z}[[t]]$ there is an equality*

$$\mathrm{sign}\,\mathrm{dlog}\,\zeta^{\mathbb{A}^1}_{X,\varphi} = \frac{d}{dt}\log\prod_i \left(P_{\varphi(\mathbb{R})|H^i_{\mathrm{top}}}(t)\right)^{(-1)^{i+1}}$$

where $P_{\varphi(\mathbb{R})|H^i_{\mathrm{top}}}(t)$ denotes the characteristic polynomial

$$P_{\varphi(\mathbb{R})|H^i}(t) = \det(1 - t\varphi(\mathbb{R})|H^i_{\mathrm{top}}(X(\mathbb{R});\mathbb{Z}))$$

of the action of $\varphi(\mathbb{R})$ on the singular cohomology of $X(\mathbb{R})$.

Proof. Let $C^*_{top} : H \to D(\mathbb{Z})$ denote the singular cochain functor from the homotopy category of topological spaces to the derived category of $\mathbb{Z}$-modules, which is symmetric monoidal. Since $D(\mathbb{Z})$ admits a tensor inverse to $C^*_{top}(S^1)$, there is an induced symmetric monoidal functor $C^*_{top} : SW \to D(\mathbb{Z})$ from the topological Spanier–Whitehead category (see Proposition 2.6). We compute

$$\begin{aligned}
\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi} &= \sum_{m\geq 1} \operatorname{Tr}(\varphi(\mathbb{R})^m)t^{m-1} && \text{(by Lemma 7.1)} \\
&= \sum_{m\geq 1} C^*_{top}\operatorname{Tr}(\varphi(\mathbb{R})^m)t^{m-1} && (C^*_{top} : \operatorname{End}(1_{SW}) \to \mathbb{Z} \text{ is an isomorphism}) \\
&= \sum_{m\geq 1} \operatorname{Tr}(C^*_{top}(\varphi(\mathbb{R})^m))t^{m-1} && (C^*_{top} \text{ is symmetric monoidal}) \\
&= \sum_{m\geq 1} \operatorname{Tr}(H^*_{top}(\varphi(\mathbb{R})^m))t^{m-1} \\
&= \sum_{m\geq 1} \sum_{i=-\infty}^{\infty} (-1)^i \operatorname{Tr}(H^i_{top}(\varphi(\mathbb{R})^m))t^{m-1} \\
&= \sum_{i=-\infty}^{\infty} -(-1)^i \frac{d}{dt} \log P_{\varphi(\mathbb{R})|H^i_{top}}(t) && \text{(by Lemma 5.6).} \qquad \square
\end{aligned}$$

For a $\mathbb{Z}[1/d]$ scheme $\mathcal{X}$ and a point $z : \operatorname{Spec} L \to \operatorname{Spec} \mathbb{Z}[1/d]$, let $\mathcal{X}_L$ denote the pullback $X_L := \mathcal{X} \otimes L$ of $\mathcal{X}$ along z. Similarly, given an endomorphism $\varphi : \mathcal{X} \to \mathcal{X}$, let $\varphi_L : \mathcal{X}_L \to \mathcal{X}_L$ denote the corresponding pullback.

Note there are pullback map on Grothendieck–Witt groups

$$z^*\colon \operatorname{GW}(\mathbb{Z}[1/d]) \to \operatorname{GW}(\mathbb{F}_q) \qquad \text{and} \qquad \eta^*_{\mathbb{R}}\colon \operatorname{GW}(\mathbb{Z}[1/d]) \to \operatorname{GW}(\mathbb{R}).$$

Moreover, for $d = 1$, the map $\eta^*_{\mathbb{R}}$ is the isomorphism $\operatorname{GW}(\mathbb{Z}) \xrightarrow{\cong} \operatorname{GW}(\mathbb{R})$, which allows us to map elements of $\operatorname{GW}(\mathbb{R})$ into $\operatorname{GW}(\mathbb{Z}[1/d])$. Let $\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d]$ denote the polynomial ring in infinitely many variables over $\mathbb{Z}$, where each of $\langle -1\rangle$ and $\langle p\rangle$ are viewed as variables. The notation defines an evident map $\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d] \to \operatorname{GW}(\mathbb{Z}[1/d])$, which is a surjection by [BW20, Lemma 5.6]. By a slight abuse of notation, we will also apply the maps z^* and $\eta^*_{\mathbb{R}}$ to elements of the ring $\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d]$.

Proposition 7.3. *Let $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}[1/d]$ be smooth and proper for $d = 1$ or d even and let $\varphi : \mathcal{X} \to \mathcal{X}$ be an endomorphism. Let z be a closed point of $\mathbb{Z}[1/d]$ with residue field $\mathbb{F}_p$. Then there is a a power series ζ in $\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d][[t]]$ such that*

$$\eta^*_{\mathbb{R}}\zeta = \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathcal{X}_{\mathbb{R}},\varphi_{\mathbb{R}}} \qquad \textit{and} \qquad z^*\zeta = \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathcal{X}_{\mathbb{F}_p},\varphi_{\mathbb{F}_p}}.$$

Proof. The scheme $\mathcal{X}$ is dualizable in $\operatorname{SH}(\mathbb{Z}[1/d])$ by [DDØ22, Theorem 3.4.2]. For d even, we have a Hermitian K-theory spectrum $\operatorname{KO} \in \operatorname{SH}(\mathbb{Z}[1/d])$ with $[1_{\mathbb{Z}[1/d]}, \operatorname{KO}] \cong \operatorname{GW}(\mathbb{Z}[1/d])$ [Hor05], and $f^*\operatorname{KO} = \operatorname{KO}$ for $f = z^*, \eta^*_{\mathbb{R}}$(and more generally) [PW18]. Let $u : 1_{\mathbb{Z}[1/d]} \to \operatorname{KO}$ denote the unit map of the ring spectrum KO. Then $u \wedge \operatorname{Tr}(\varphi^m) \in [1_{\mathbb{Z}[1/d],\operatorname{KO}}] \cong \operatorname{GW}(\mathbb{Z}[1/d])$. Choose preimages $(\widetilde{u \wedge \operatorname{Tr}(\varphi^m)})$ in $\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d]$ of $u \wedge \operatorname{Tr}(\varphi^m)$ under the surjection

$$\mathbb{Z}[\langle -1\rangle, \langle p\rangle : p \text{ prime dividing } d] \to \operatorname{GW}(\mathbb{Z}[1/d]).$$

Define ζ by

$$\zeta := \sum_{m=1}^{\infty} (u \wedge \widetilde{\mathrm{Tr}(\varphi^m)}) t^{m-1}$$

Let $f : \operatorname{Spec} L \to \operatorname{Spec} \mathbb{Z}[1/d]$ denote either of the maps $z : \operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}[1/d]$ or $\eta_{\mathbb{R}} : \operatorname{Spec} \mathbb{R} \to \operatorname{Spec} \mathbb{Z}[1/d]$. The proposition then follows from the following equalities in $\mathrm{GW}(L)$:

$$\begin{aligned} f^*(u \wedge \mathrm{Tr}(\varphi^m)) &= f^* u \wedge f^*(\mathrm{Tr}(\varphi^m)) && (f^* \text{ is symmetric monoidal}) \\ &= u \wedge \mathrm{Tr}(f^*\varphi^m) && (f^*\mathrm{KO} \simeq \mathrm{KO} \text{ and Proposition 2.4}) \\ &= \mathrm{Tr}(f^*\varphi^m) && ([1_L, 1_L] \xrightarrow{\cong} [1_L, \mathrm{KO}]). \end{aligned}$$ □

Corollary 7.4. *For $\varphi : \mathcal{X} \to \mathcal{X}$ an endomorphism of a smooth and proper $\mathbb{Z}$-scheme.*

$$\operatorname{dlog}^{\mathbb{A}^1} \zeta_{\mathcal{X}_{\mathbb{F}_q}, \varphi_{\mathbb{F}_q}} = z^* \operatorname{dlog}^{\mathbb{A}^1} \zeta_{\mathcal{X}_{\mathbb{R}}, \varphi_{\mathbb{R}}}.$$

Proof. Follows from Proposition 7.3 and the isomorphism $\eta_{\mathbb{R}}^* : \mathrm{GW}(\mathbb{Z}) \xrightarrow{\cong} \mathrm{GW}(\mathbb{R})$. □

Example 7.5. If $\varphi : X \to X$ is the relative Frobenius of a smooth proper toric variety X over $\mathbb{F}_p$, the Frobenius lifts to $\varphi : \mathcal{X} \to \mathcal{X}$ with $\mathcal{X}$ smooth and proper over $\mathbb{Z}$ (see, e.g., [BTLM97, Section 3.4] or [Bor09, Section 2.4]). (The criteria for smoothness and properness of toric varieties shows that the lift $\mathcal{X}$ given in the references is indeed smooth and proper.) Combining Corollary 7.4 with Proposition 7.2 and (1) computes the logarithmic $\mathbb{A}^1$-zeta function of these varieties.

When the Frobenius does not lift, we wish to use Theorems 6.1 and 6.2 to relate the logarithmic $\mathbb{A}^1$-zeta function to the real points of a lift over a ring with a real place.

Let X be a strictly cellular scheme over $\mathbb{R}$. By Remark 4.1, the complex $C_*^{\mathrm{cell}}(X)$ is quasi-isomorphic to a complex of the form

$$(16) \qquad \ldots \leftarrow 0 \leftarrow \mathbb{Z}^{b_0} \leftarrow (K_1^{MW})^{b_1} \leftarrow (K_2^{MW})^{b_2} \leftarrow \ldots \leftarrow (K_n^{MW})^{b_n} \leftarrow 0 \leftarrow \ldots$$

Proposition 7.6. *Let X be a strictly cellular scheme over $\mathbb{R}$ and let b_i be as in (16). Then there is a complex C_* in $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ with $C_i \cong \mathbb{Z}^{b_i}$ and a quasi-isomorphism*

$$C_* \simeq C_*^{\mathrm{top}}(\mathcal{X}(\mathbb{R}); \mathbb{Z})$$

between C_* and the singular chains on the real manifold $X(\mathbb{R})$.

Proof. Let $\emptyset = \Omega_{-1}(X) \subset \Omega_0(X) \subset \Omega_1(X) \subset \cdots \subset \Omega_{n-1}(X) \subset \Omega_n(X) = X$ denote a strictly cellular structure on X, in the sense of Definition 3.1. So $\Omega_i(X) \subset X$ is an open subscheme and

$$\Omega_i(X) \smallsetminus \Omega_{i-1}(X) \cong \coprod_{\alpha \in \beta_i} \mathbb{A}^{n-i}$$

is a finite disjoint union of schemes isomorphic to affine space $\mathbb{A}^{n-i}$ indexed by the set β_i. By Remark 4.1, the cardinality $|\beta_i| = b_i$ of β_i is b_i. For each i, there is a cofiber sequence of topological spaces

$$\Omega_{i-1}(X)(\mathbb{R}) \to \Omega_i(X)(\mathbb{R}) \to \Omega_i(X)(\mathbb{R})/\Omega_{i-1}(X)(\mathbb{R}) \simeq \vee_{\alpha \in \beta_i} \mathrm{Th}_{\mathbb{A}^{n-i}(\mathbb{R})} \nu_\alpha$$

where ν_α denotes the normal bundle of the component of $\Omega_i(X) \smallsetminus \Omega_{i-1}(X)$ indexed by α in $\Omega_i(X)(\mathbb{R})$. Since $\mathbb{A}^{n-i}(\mathbb{R})$ is contractible, we may choose weak equivalences $\mathrm{Th}_{\mathbb{A}^{n-i}(\mathbb{R})} \nu_\alpha \simeq S^i$. Applying singular cochains C_*^{top} gives a distinguished triangle

$$C_*^{\mathrm{top}}(\Omega_{i-1}(X)(\mathbb{R})) \to C_*^{\mathrm{top}}(\Omega_i(X)(\mathbb{R})) \to \tilde{C}_*^{\mathrm{top}}(\vee_{\beta_i} S^i)$$

in $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$, where $\tilde{C}_*^{\mathrm{top}}$ denote the reduced singular chains. The reduced singular chains on the wedge are quasi-isomorphic to a complex concentrated in degree i, namely

$$\tilde{C}_*^{\mathrm{top}}(\vee_{\beta_i} S^i) \simeq \mathbb{Z}^{b_i}[i].$$

The claim follows by induction on i. □

Note that the definition of a strict cellular structure (see Definition 3.1) makes sense over an arbitrary base scheme. Given a smooth proper strictly cellular scheme $\mathcal{X} \to \operatorname{Spec} A$ over a ring A and a real point $\eta : \operatorname{Spec} \mathbb{R} \to A$, define $\mathcal{X}_\eta(\mathbb{R})$ to be the real manifold associated to the real points of $\mathcal{X} \times_A \mathbb{R}$.

Proposition 7.7. *Suppose $\mathcal{X} \to \operatorname{Spec} A$ is a smooth projective strictly cellular scheme with points $z : \operatorname{Spec} \mathbb{F}_q \to A$ and $\eta : \operatorname{Spec} \mathbb{R} \to A$. Let $X := \mathcal{X} \times_A \mathbb{F}_q$ denote the pullback of $\mathcal{X}$ along z. Let $\varphi : X \to X$ denote an endomorphism of X. Then*

(1) *There is a complex C_* in $\mathbf{Ch}_{\geq 0}(\mathbb{Z})$ of the form*

$$\ldots \leftarrow 0 \leftarrow \mathbb{Z}^{b_0} \leftarrow \mathbb{Z}^{b_1} \leftarrow \mathbb{Z}^{b_2} \leftarrow \ldots \leftarrow \mathbb{Z}^{b_n} \leftarrow 0 \leftarrow \ldots$$

which is quasi-isomorphic to the topological chains on $\mathcal{X}_\eta(\mathbb{R})$

$$C_* \simeq C_*^{\mathrm{top}}(\mathcal{X}_\eta(\mathbb{R}); \mathbb{Z}).$$

(2) *The $\mathbb{A}^1$-logarithmic zeta function of φ is given by the formula*

$$\operatorname{dlog} \zeta_{X,\varphi}^{\mathbb{A}^1} = \sum_{i=-\infty}^{\infty} -\langle -1 \rangle^i \frac{d}{dt} \log P_{C_i^{\mathrm{cell}}(\varphi)}(t)$$

where

$$P_{C_i^{\mathrm{cell}}(\varphi)}(t) = \det(1 - tC_i^{\mathrm{cell}}(\varphi)).$$

and $C_i^{\mathrm{cell}}(\varphi)$ is a square matrix of elements of $\mathrm{GW}(\mathbb{F}_q)$ of size $b_i \times b_i$.

Proof. Let C_* denote the complex of Proposition 7.6 applied to the strictly cellular scheme $\mathcal{X} \times_\eta \mathbb{R}$ over $\mathbb{R}$. The condition of (1) is satisfied by Proposition 7.6. Moreover, b_i is the cardinality of β_i, which is defined to be the set of connected components of $\Omega_i(X) \smallsetminus \Omega_{i-1}(X)$ in the decomposition

$$\Omega_i(X) \smallsetminus \Omega_{i-1}(X) \cong \coprod_{\alpha \in \beta_i} \mathbb{A}^{n-i}.$$

The formula (2) then follows from Theorem 6.2. □

8. Computing $\mathbb{A}^1$-logarithmic zeta functions and examples

In this section, we describe two methods to explicitly compute examples of $\mathbb{A}^1$-zeta functions. First, for schemes with a strict cellular structure over a finite field, we explicitly compute the enriched logarithmic zeta function of Frobenius endomorphisms using Theorem 6.2; this gives examples of the enriched logarithmic zeta function for varieties like projective space and Grassmannians.

Then over a finite field, we also use Hoyois' enriched Grothendieck–Lefschetz trace formula and Möbius inversion to obtain the coefficients of the enriched logarithmic zeta function of the Frobenius from those of the logarithmic derivative of the classical zeta function. We may use this method to compute the enriched logarithmic zeta function for non-cellular schemes, such as elliptic curves.

8.1. $\mathbb{A}^1$-zeta function of Frobenius endomorphisms using Theorem 6.2.

Let k be a finite field $\mathbb{F}_q$ and X be a smooth proper k-scheme with a strict cellular structure

$$\emptyset = \Sigma_{-1}(X) \subset \Sigma_0(X) \subset \Sigma_1(X) \subset \cdots \subset \Sigma_{n-1}(X) \subset \Sigma_n(X) = X \tag{17}$$

defined over k. As above, we set $\Omega_i(X) := X \setminus \Sigma_{n-i-1}(X)$.

Following [Mor12, Lemma 3.14], for a positive integer n, let $n_\epsilon \in \mathrm{GW}(k)$ denote

$$n_\epsilon := \langle 1 \rangle + \langle -1 \rangle + \langle 1 \rangle + \ldots + \langle -1 \rangle,$$

the class of the rank n diagonal form with alternating 1's and -1's along the diagonal. Note that $n_\epsilon m_\epsilon = (nm)_\epsilon$.

Proposition 8.1. *Let X be a smooth proper scheme over $\mathbb{F}_q$ with a strict cellular structure defined over $\mathbb{F}_q$, and let φ be the relative Frobenius. Then $C^{\mathrm{cell}}_*(\varphi)$ is the map of complexes which in degree i*

$$C^{\mathrm{cell}}_i(\varphi) : C^{\mathrm{cell}}_i(X) \to C^{\mathrm{cell}}_i(X)$$

is multiplication by q^i_ϵ.

Proof. We use the above notation for the strict cellular structure (17) on X. Since the relative Frobenius φ gives an endomorphism of the smooth pair $(\Omega_i(X), \Omega_i(X) \setminus X_i)$, it determines a map $\varphi : \mathrm{Th}_{X_n}(\nu_n) \to \mathrm{Th}_{X_n}(\nu_n)$ by purity [MV99, Theorem 2.23]. The map of complexes $C^{\mathrm{cell}}_*(\varphi)$ in degree i is given by $\widetilde{\mathrm{H}}^{\mathbb{A}^1}_i(\varphi) : \widetilde{\mathrm{H}}^{\mathbb{A}^1}_i(\mathrm{Th}_{X_i}(\nu_i)) \to \widetilde{\mathrm{H}}^{\mathbb{A}^1}_i(\mathrm{Th}_{X_i}(\nu_i))$.

For each $m \in M_i$, choose an $\mathbb{F}_q$-point p_m in the corresponding connected component of $X_i \cong \coprod_{m \in M_i} X_{im}$ and $X_{im} \cong \mathbb{A}^{n-i}$. The inclusion

$$\mathrm{Th}_{p_m} \nu_i \hookrightarrow \mathrm{Th}_{X_{im}} \nu_i$$

is an $\mathbb{A}^1$-weak equivalence.

Since X has a strict cellular structure, the Krull dimension of X is n. By [sga03, Théorème II.4.10], we may choose a Zariski open subset $U \subset X$ containing p_m and an étale map $\psi : U \to \mathbb{A}^n$ such that

$$\begin{array}{ccc} X_{im} \cap U & \longrightarrow & \Omega_i(X) \cap U \\ \downarrow & & \downarrow \psi \\ \mathbb{A}^{n-i} & \longrightarrow & \mathbb{A}^n \end{array}$$

is a pullback, and the bottom horizontal map is the natural map $\operatorname{Spec} k[y_{i+1}, \ldots, y_n] \to \operatorname{Spec} k[y_1, \ldots, y_n]$ given by $(y_{i+1}, \ldots, y_n) \mapsto (0, \ldots, 0, y_{i+1}, \ldots, y_n)$. We obtain maps of Thom spaces

$$\mathrm{Th}_{p_m} \nu_i \to \mathrm{Th}_{X_{im} \cap U} \nu_i \to \mathrm{Th}_{\mathbb{A}^{n-i}} \mathsf{N}_{\mathbb{A}^{n-i}} \mathbb{A}^n$$

whose composition is an $\mathbb{A}^1$-weak equivalence.

Purity [MV99, Theorem 2.23] defines a canonical $\mathbb{A}^1$-weak equivalence

$$\mathrm{Th}_{\mathbb{A}^{n-i}} \mathsf{N}_{\mathbb{A}^{n-i}} \mathbb{A}^n \simeq \mathbb{A}^n / \mathbb{A}^n - \mathbb{A}^{n-i}.$$

The map $\operatorname{Spec} k[y_1, \ldots, y_i] \to \operatorname{Spec} k[y_1, \ldots, y_n]$ given by $(y_1, \ldots, y_i) \mapsto (y_1, \ldots, y_i, 0, \ldots, 0)$ determines an $\mathbb{A}^1$-weak equivalence

$$\mathbb{A}^i / \mathbb{A}^i - \mathbb{A}^0 \simeq \mathbb{A}^n / \mathbb{A}^n - \mathbb{A}^{n-i}$$

and excision determines an $\mathbb{A}^1$-weak equivalence

$$\mathbb{A}^i / \mathbb{A}^i - \mathbb{A}^0 \simeq \mathbb{P}^i / \mathbb{P}^{i-1}.$$

Composing in the homotopy category, we obtain a canonical (zig-zag) $\mathbb{A}^1$-weak equivalence

$$\mathrm{Th}_{X_i}(\nu_i) \simeq \mathbb{P}^i / \mathbb{P}^{i-1}$$

where the relative Frobenius acts compatibly on both sides. The claim is thus equivalent to showing that the map

$$\widetilde{H}_i^{\mathbb{A}^1}(\varphi) : \widetilde{H}_i^{\mathbb{A}^1}(\mathbb{P}^i / \mathbb{P}^{i-1}) \to \widetilde{H}_i^{\mathbb{A}^1}(\mathbb{P}^i / \mathbb{P}^{i-1})$$

induced by φ is multiplication by q_ϵ^i. By Morel's Hurewicz theorem, this map $\widetilde{H}_i^{\mathbb{A}^1}(\varphi)$ is multiplication by $\deg^{\mathbb{A}^1} \varphi$. The relative Frobenius is the map

$$\mathbb{P}^i / \mathbb{P}^{i-1} \to \mathbb{P}^i / \mathbb{P}^{i-1}$$
$$y_j \mapsto y_j^q$$

for $j = 0, \ldots, i$, which is homotopy equivalent to the i-fold smash product of $\mathbb{P}^1 \to \mathbb{P}^1$ mapping $y_j \mapsto y_j^q$. The degree of this map is q_ϵ^i. (To see this, first note that the smash product multiplies degrees [Mor04] so it suffices to prove the claim for $i = 1$. The computation for $i = 1$ follows in a straightforward manner from computing the Bézoutian [Caz12, Definition 3.4] which computes the $\mathbb{A}^1$-degree [Caz12, Theorem 3.6 and Theorem 1.2].) The degree and the claim follows. □

Using Proposition 8.1 we can give an explicit description of the enriched logarithmic zeta function of a smooth projective scheme over $\mathbb{F}_q$ equipped with an $\mathbb{F}_q$-rational strict cellular structure. Recall that in this case, for each i,

$$C_i^{\mathrm{cell}}(X) \simeq (\underline{K}_i^{MW})^{b_i}$$

for a nonnegative integer $b_i \geq 0$. We call the integer b_i the rank of $C_i^{\mathrm{cell}}(X)$.

Corollary 8.2. *Let X be a smooth projective scheme of dimension n over $\mathbb{F}_q$ equipped with a strict cellular structure defined over $\mathbb{F}_q$. Let b_i be the rank of $C_i^{\mathrm{cell}}(X)$. Then*

$$\operatorname{dlog} \zeta_X^{\mathbb{A}^1}(t) = \sum_{i=0}^{n} -\langle -1 \rangle^i \frac{d}{dt} \log(1 - q_\epsilon^i t)^{b_i}.$$

Proof. Let φ denote the relative Frobenius endomorphism. By Theorem 6.2, it suffices to show that $P_{C_i^{\mathrm{cell}}(\varphi)}(t) = (1 - q_\epsilon^i t)^{b_i}$. This follows from the fact that $C_i^{\mathrm{cell}}(\varphi)$ is a $b_i \times b_i$ square matrix, which by Proposition 8.1 is multiplication by q_ϵ^i. □

Remark 8.3. We think of Corollary 8.2 as morally saying

$$\text{“ } \zeta_X^{\mathbb{A}^1}(t) = \frac{1}{\prod_{i \text{ odd}}(1-q_\epsilon^i)^{b_i\langle -1\rangle}\prod_{i \text{ even}}(1-q_\epsilon^i)^{b_i}} \text{ ”}.$$

As explained in Remark 6.3, this does not make literal sense because there is not a λ-ring structure on $\mathrm{GW}(\mathbb{F}_q)(t)$ compatible with the logarithmic derivative.

Corollary 8.2 yields many examples of enriched logarithmic zeta functions for varieties of natural geometric interest.

Example 8.4 (Enriched logarithmic zeta function of $\mathbb{P}^n$)**.** Projective space $\mathbb{P}^n$ has a strict cellular structure with a single copy of $\mathbb{A}^i$ for each $0 \le i \le n$. Applying Corollary 8.2 shows that the logarithmic zeta function of $\mathbb{P}^n$ is given by

$$\mathrm{dlog}\, \zeta_{\mathbb{P}^n}^{\mathbb{A}^1}(t) = \frac{d}{dt}\log \frac{1}{\prod\limits_{\substack{0\le i\le n\\ i \text{ even}}}(1-q_\epsilon^i t)} + \langle -1\rangle \frac{d}{dt}\log\frac{1}{\prod\limits_{\substack{1\le i\le n\\ i \text{ odd}}}(1-q_\epsilon^i t)}.$$

Remark 8.5 (Functional equation for projective spaces)**.** We record here that the $\mathbb{A}^1$-logarithmic zeta function of projective spaces of odd dimension satisfies a functional equation.

When n is odd, we have

$$\chi_c^{\mathbb{A}^1}(\mathbb{P}^n) = \frac{n+1}{2}(\langle 1\rangle + \langle -1\rangle)$$

(see [Lev20, Example 1.6]). By Example 8.4, we have

$$\frac{d}{dt}\log\zeta_{\mathbb{P}^n}^{\mathbb{A}^1}(t) = \frac{d}{dt}\log\frac{1}{\prod_{\substack{0\le i\le n\\ i \text{ even}}}(1-\tilde{q}^i t)} + \langle -1\rangle\frac{d}{dt}\log\frac{1}{\prod_{\substack{1\le i\le n\\ i \text{ odd}}}(1-\tilde{q}^i t)}.$$

For any i such that $0 \le i \le n$, we have

$$\frac{1}{1-\tilde{q}^i\frac{1}{\tilde{q}^n t}} = \frac{\tilde{q}^n t}{\tilde{q}^i(\tilde{q}^{n-i}t-1)} = \frac{-\tilde{q}^{n-i}t}{1-\tilde{q}^{n-i}t}.$$

Combining the last three equations, we see that

$$\frac{d}{dt}\log\zeta_{\mathbb{P}^n}^{\mathbb{A}^1}(t) = \chi_c^{\mathbb{A}^1}(\mathbb{P}^n)t + \frac{d}{dt}\log\zeta_{\mathbb{P}^n}^{\mathbb{A}^1}\left(\frac{1}{\tilde{q}^n t}\right).$$

The functional equation for the regular zeta function is a consequence of Poincare duality for the ℓ-adic étale cohomology groups. Morel and Sawant have interesting conjectures on Poincare duality for their cohomology theories for general smooth projective varieties X. It would be interesting to know if an analogous functional equation of the logarithmic zeta function can be deduced from such a statement for general smooth projective varieties.

Example 8.6 (Enriched logarithmic zeta function of $\mathbb{G}(1,3)$)**.** The Grassmannian $\mathbb{G}(1,3)$ of lines in $\mathbb{P}^3$ has a strict cellular structure composed of the Schubert cycles, as we now recall. Fix a full flag $V_0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{P}^3$ of linear subspaces, where V_i has dimension i. For $0 \le b \le a \le 2$, let $\Sigma_{a,b}$ denote the locus of lines in $\mathbb{G}(1,3)$ that meet V_{2-a} in a point and V_{3-b} in a line. Let $\Sigma_{a,b}^\circ$ denote the complement of all other Schubert cycles in $\Sigma_{a,b}$. Then, as in [EH16, Section 3.3.1], we have $\Sigma_{a,b} \simeq \mathbb{A}^{4-a-b}$, and the filtration by closed subsets

$$\emptyset \subset \Sigma_{2,2} \subset \Sigma_{2,1} \subset (\Sigma_{1,1}\cup\Sigma_{2,0}) \subset \Sigma_{1,0} \subset \Sigma_{0,0} = \mathbb{G}(1,3)$$

gives a strict cellular structure as in Definition 3.1(1). Hence we have $b_2 = 2$ and all other $b_i = 1$ for $i = 0, 1, 3, 4$. Thus

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{G}(1,3)}(t) = \frac{d}{dt} \log \frac{1}{(1-t)(1-q_\epsilon^2 t)^2(1-q_\epsilon^4 t)} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{(1-q_\epsilon t)(1-q_\epsilon^3 t)}.$$

Example 8.7 (Enriched logarithmic zeta function of $\mathbb{P}^1 \times \mathbb{P}^1$)**.** The product $\mathbb{P}^1 \times \mathbb{P}^1$ has a strict cellular structure with a single copy of $\mathbb{A}^0$, two copies of $\mathbb{A}^1$ and one copy of $\mathbb{A}^2$. Applying Corollary 8.2 shows that the logarithmic zeta function of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{P}^1 \times \mathbb{P}^1}(t) = \frac{d}{dt} \log \frac{1}{(1-t)(1-q_\epsilon^2 t)} + \langle -1 \rangle \frac{d}{dt} \log \frac{1}{(1-q_\epsilon t)^2}.$$

8.2. **$\mathbb{A}^1$-logarithmic zeta functions via Hoyois's trace formula.** In [Hoy15, Theorem 1.3], Hoyois provides a quadratic refinement of the Grothendieck–Lefschetz trace formula in the setting of stable motivic homotopy theory, which gives us a procedure to compute the coefficients of the enriched logarithmic zeta function $\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi}$ from those of $\operatorname{dlog} \zeta_X(t)$.

We introduce some notation first. For an endomorphism $\varphi : X \to X$ of a scheme X, we denote by X^φ its scheme of fixed points. Given a finite separable field extension L/K, the classical trace map $\operatorname{Tr}_{L/K} : L \to K$ induces a *Transfer*

$$\operatorname{Tr}_{L/K} : \mathrm{GW}(L) \to \mathrm{GW}(K) \tag{18}$$

by sending a bilinear form $b : V \times V \to L$ to the form $\operatorname{Tr}_{L/K} \circ b : V \times V \to K$ of rank $[L : K] \operatorname{rank}(b)$ (see [Mor12, The cohomological transfer, Chapter 4] [CF17, Lemma 2.3]).

Hoyois' main theorem then has the following consequence:

Proposition 8.8. [Hoy15, Corollary 1.10] *Let k be a field, let X be a smooth proper k-scheme, and let $\varphi : X \to X$ be a k-morphism with étale fixed points. Then*

$$\operatorname{Tr}(\varphi) = \sum_{x \in X^\varphi} \operatorname{Tr}_{\kappa(x)/k} \langle \det(\mathrm{Id} - d\varphi_x) \rangle.$$

This result gives a computation of the $\mathbb{A}^1$-logarithmic zeta function of a smooth proper scheme over a finite field in terms of its point counts.

Theorem 8.9. *Let X be a smooth proper scheme over $\mathbb{F}_q$ and let $\varphi : X \to X$ be the relative Frobenius morphism. Let u denote a non-square in $\mathbb{F}_q$. Then the $\mathbb{A}^1$-logarithmic zeta function $\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi}$ is computed by the following formula:*

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi} = \sum_m \left(\left(\sum_{\substack{i \mid m \\ i \text{ even}}} \left(\frac{1}{i} \sum_{d \mid i} \mu(d) |X(\mathbb{F}_{q^{i/d}})| \right) (i-1)\langle 1 \rangle + \langle u \rangle \right) \right.$$
$$\left. + \left(\sum_{\substack{i \mid m \\ i \text{ odd}}} \left(\frac{1}{i} \sum_{d \mid i} \mu(d) |X(\mathbb{F}_{q^{i/d}})| \right) i \langle 1 \rangle \right) \right) t^{m-1}.$$

Proof. For X a smooth proper scheme over $\mathbb{F}_q$ and $\varphi : X \to X$ the Frobenius morphism, we apply Hoyois' formula to φ^m. We can write X^{φ^m} as a disjoint union over all of the points of degree X of degree dividing m:

$$X^{\varphi^m} = \coprod_{i|m} \coprod_{\substack{\text{degree } i \\ \text{points of } X}} \operatorname{Spec} \mathbb{F}_{q^i}. \tag{19}$$

Denote by $\alpha(i)$ the number of points of degree i on X. Since $d\varphi^m = 0$, we obtain that

$$N_m(X) := \operatorname{Tr}(\varphi^m) = \sum_{i|m} \alpha(i) \operatorname{Tr}_{\mathbb{F}_{q^i}/\mathbb{F}_q} \langle 1 \rangle. \tag{20}$$

It is classical that

$$\operatorname{Tr}_{\mathbb{F}_{q^i}/\mathbb{F}_q} \langle 1 \rangle = \begin{cases} i\langle 1 \rangle & \text{if } i \text{ odd} \\ (i-1)\langle 1 \rangle + \langle u \rangle & \text{if } i \text{ even} \end{cases} \tag{21}$$

(see e.g. Lemma 58 in [KW21]). It remains to show that

$$\alpha(i) = \frac{1}{i} \sum_{d|i} \mu(d) |X(\mathbb{F}_{q^{i/d}})|, \tag{22}$$

This follows from Möbius inversion since for every $i \geq 1$, we have

$$|X(\mathbb{F}_{q^i})| = \sum_{d|i} \alpha(d) d.$$

□

Remark 8.10. The quantity $|X(\mathbb{F}_{q^{i/d}})|$ in Theorem 8.9 can be computed from the eigenvalues of the Frobenius morphism on étale cohomology groups of X. More precisely, choose a prime ℓ coprime to q. For i such that $0 \leq i \leq 2 \dim X$, let $\{\lambda_{i,1}, \ldots, \lambda_{i,b_i}\}$ denote the eigenvalues of the Frobenius morphism on $H^i_{\text{ét}}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$. Then the Grothendieck–Lefschetz trace formula ([Poo17, Theorem 7.1.1(ii), Section 7.5.7]) tells us that

$$|X(\mathbb{F}_{q^{i/d}})| = \sum_{j=0}^{2 \dim X} (-1)^j \left(\sum_{l=1}^{b_j} \lambda_{j,l}^{i/d} \right).$$

Remark 8.11. From (21) and (22), we see that only the case where q is odd will be interesting, and that the only contributions to $\operatorname{disc} N_m(X)$ come from $i \mid m$ with i even. In particular, $\operatorname{disc} N_m(X)$ is trivial for all odd m and we have the expression $N_m(X) = |X(\mathbb{F}_{q^m})| \langle 1 \rangle$ for all odd m.

For even m, we have the formula

$$\operatorname{disc} N_m(X) = \sum_{\substack{i|m \\ i \text{ even}}} \alpha(i) \tag{23}$$

in $\mathbb{Z}/2\mathbb{Z}$, and therefore, using (22) again, that

$$\operatorname{disc} N_m(X) = \sum_{\substack{i|m \\ i \text{ even}}} \frac{1}{i} \sum_{d|i} \mu(d) |X(\mathbb{F}_{q^{i/d}})|. \tag{24}$$

Example 8.12 (Enriched logarithmic zeta function of $\operatorname{Spec} \mathbb{F}_{q^2}$)**.** Since $\operatorname{Spec} \mathbb{F}_{q^2}$ has exactly one closed point of degree 2, it follows from Theorem 8.9 that

$$N_m(\operatorname{Spec} \mathbb{F}_{q^2}) = \begin{cases} 0 & \text{if } m \text{ odd} \\ \langle 1 \rangle + \langle u \rangle & \text{if } m \text{ even} \end{cases},$$

and hence

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{\operatorname{Spec}\mathbb{F}_{q^2},\varphi} = \sum_{m \text{ even}} (\langle 1\rangle + \langle u\rangle) t^{m-1} = \langle u\rangle \operatorname{dlog} \frac{1}{1-\langle u\rangle t} + \operatorname{dlog} \frac{1}{1+t}.$$

Example 8.13 (Enriched logarithmic zeta function of a twisted product of the projective line)**.** Let $X := \operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$. The difference between the cellular structure of $\mathbb{P}^1 \times \mathbb{P}^1$ as in Example 8.7 and that of X is that $\Sigma_1(X) \setminus \Sigma_0(X)$ is $\mathbb{A}^1_q \coprod \mathbb{A}^1_q$ for $\mathbb{P}^1 \times \mathbb{P}^1$ whereas it is $\mathbb{A}^1 \times_{\mathbb{F}_q} \operatorname{Spec}\mathbb{F}_{q^2}$ for X.

We will prove that

$$N_m(X) = \begin{cases} 1 + q_\epsilon^{2m} & \text{if } m \text{ odd} \\ 1 + q_\epsilon^{2m} + (\langle 1\rangle + \langle u\rangle)(-\langle -1\rangle q_\epsilon^m) & \text{if } m \text{ even.} \end{cases}$$

Once we have this, a direct calculation will then show that

$$\operatorname{dlog} \zeta^{\mathbb{A}^1}_{X,\varphi} = \frac{d}{dt} \log \frac{1}{(1-t)(1-q_\epsilon^2 t)} + \langle -u\rangle \frac{d}{dt} \log \frac{1}{1-q_\epsilon \langle u\rangle t} + \langle -1\rangle \frac{d}{dt} \log \frac{1}{1+q_\epsilon t}.$$

Since $\mathbb{P}^1 = \operatorname{Spec}\mathbb{F}_q \coprod \mathbb{A}^1$, $N_m(\mathbb{P}^1) = 1 - \langle -1\rangle q_\epsilon^m$, $N_m(\operatorname{Spec}\mathbb{F}_q) = 1$ (Example 8.4) and N_m is a motivic measure (Proposition 9.1), it follows that $N_m(\mathbb{A}^1) = -\langle -1\rangle q_\epsilon^m$. Once again using the fact that N_m is a motivic measure, we get that

$$N_m(\mathbb{A}^1 \times \operatorname{Spec}\mathbb{F}_{q^2}) = N_m(\mathbb{A}^1) \times N_m(\operatorname{Spec}\mathbb{F}_{q^2}) = \begin{cases} 0 & \text{if } m \text{ odd} \\ (\langle 1\rangle + \langle u\rangle)(-\langle -1\rangle q_\epsilon^m) & \text{if } m \text{ even} \end{cases}$$

We also have $\Sigma_2(X) \setminus \Sigma_1(X) = \operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{A}^1 \cong \mathbb{A}^2$, and $\Sigma_1(X) \setminus \Sigma_0(X) = \operatorname{Spec}\mathbb{F}_q$. Since $N_m(\mathbb{A}^2) = N_m(\mathbb{A}^1)^2$, and $X = \Sigma^2(X)$, putting the last few lines together, and once again using the fact that N_m is a motivic measure, we obtain the formula above for $N_m(X)$.

Remark 8.14. It is easy to write down equations for the variety in Example 8.13 explicitly. A smooth quadric Q in $\mathbb{P}^3$ over $\mathbb{F}_q$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{F}_q$ if the discriminant of the corresponding bilinear form is a square in $\mathbb{F}_q$ (equivalently when the two rulings are defined over $\mathbb{F}_q$), and is isomorphic to $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ otherwise (equivalently when the two rulings are defined over $\mathbb{F}_{q^2}$ but not over $\mathbb{F}_q$).

For example, the quadric with defining equation $11x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$ has nonsquare discriminant over $\mathbb{F}_3$ and is isomorphic to $\operatorname{Res}_{\mathbb{F}_9/\mathbb{F}_3} \mathbb{P}^1$, whereas it has square discriminant over $\mathbb{F}_5$ and hence is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{F}_5$.

Remark 8.15. The calculation in Example 8.7 and Example 8.13 illustrate the connection with the topology of the real points of a lift of $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ to characteristic 0 as in Proposition 7.2. As we remarked in the introduction, when q is congruent to 3 modulo 4, we have $u = -1$ and the extension $\mathbb{F}_q \subset \mathbb{F}_{q^2}$ is given by $\mathbb{F}_{q^2} = \mathbb{F}_q[\sqrt{-1}]$ and the $\mathbb{R}$-schemes $\mathbb{P}^1 \times \mathbb{P}^1$ and $\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{P}^1$ are lifts to characteristic zero of the varieties $\mathbb{P}^1 \times \mathbb{P}^1$ and $\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1$ over $\mathbb{F}_q$ respectively. These varieties have natural lifts to $\mathbb{Z}$-schemes, and the Frobenius endomorphism also lifts, so Proposition 7.3 applies with $d = 1$. The former $\mathbb{R}$-scheme has two 1-cells, whereas the latter has no 1-cells, which is illustrated in the calculation below as an additional $-\frac{d}{dt} \log \frac{1}{(1-t)^2}$ in $\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{P}^1 \times \mathbb{P}^1,\varphi}$ when compared to $\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q} \mathbb{P}^1,\varphi}$.

Since q is odd, we have $\operatorname{sign} q_\epsilon = 1$ and $\operatorname{sign} q_\epsilon^2 = 1$. Since q is congruent to 3 modulo 4, we have $u = -1$ and $\operatorname{sign}\langle u\rangle = -1$. Combining these with the calculations in Example 8.7

and Example 8.13, we get that

$$\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\mathbb{P}^1\times\mathbb{P}^1,\varphi} = \frac{d}{dt}\log\frac{1}{(1-t)(1-t)} - \frac{d}{dt}\log\frac{1}{(1-t)^2}, \qquad \text{and}$$

$$\begin{aligned}\operatorname{sign} \operatorname{dlog} \zeta^{\mathbb{A}^1}_{\operatorname{Res}_{\mathbb{F}_{q^2}/\mathbb{F}_q}\mathbb{P}^1,\varphi} &= \frac{d}{dt}\log\frac{1}{(1-t)(1-t)} + \frac{d}{dt}\log\frac{1}{1+t} + (-1)\frac{d}{dt}\log\frac{1}{1+t} \\ &= \frac{d}{dt}\log\frac{1}{(1-t)(1-t)}.\end{aligned}$$

8.3. **The logarithmic zeta function of non-cellular schemes.** Observe that Theorem 8.9 applies to any smooth projective scheme X, not necessarily cellular. One may hope to directly prove that the enriched logarithmic zeta function of any smooth projective scheme is dlog rational from the formula in Theorem 8.9, without appealing to any good underlying cohomology theory.

We illustrate Theorem 8.9 in the first interesting example of a non-cellular scheme, namely the case of an elliptic curve E. We will use the connection with the eigenvalues of the Frobenius endomorphism on the ℓ-adic étale cohomology groups as in Remark 8.10. The cohomology groups $H^0_{\text{ét}}(E_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ and $H^2_{\text{ét}}(E_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ are 1-dimensional, and the eigenvalues of the Frobenius endomorphism are 1 and q respectively, and that $H^1_{\text{ét}}(E_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell)$ is 2-dimensional with the two Frobenius eigenvalues $\lambda, \overline{\lambda}$ that satisfy $\lambda+\overline{\lambda} = a$ for some integer a and $\lambda\overline{\lambda} = q$ (see [Poo17, Section 7.2]). These imply

$$E(\mathbb{F}_q^{i/d}) = 1 - (\lambda^{i/d} + (\overline{\lambda})^{i/d}) + q^{i/d},$$

and in particular, that

$$E(\mathbb{F}_q) = 1 - a + q, \qquad \text{and,} \qquad E(\mathbb{F}_{q^2}) = 1 - (a^2 - 2q) + q^2.$$

In particular, by Theorem 8.9 the coefficient of t^1 in $\operatorname{dlog}^{\mathbb{A}^1}_E(t)$ is

$$\frac{a - a^2 + q + q^2}{2}\langle u\rangle + \frac{2 - a - a^2 + 3q + q^2}{2}\langle 1\rangle.$$

This coefficient has nontrivial discriminant if and only if $\frac{a-a^2+q+q^2}{2}$ is odd, or equivalently, when $q \equiv 3 \mod 4$ and $a \equiv 2, 3 \mod 4$, and similarly when $q \equiv 1 \mod 4$ and $a \equiv 0, 1 \mod 4$.

Continuing this way, for the elliptic curve with Weierstrass equation $y^2 = x^3 + 2x + 3$ over $\mathbb{F}_7$, which has $a = 2$, we find

$$\begin{aligned}\operatorname{dlog}^{\mathbb{A}^1}_E(t) = 6\langle 1\rangle t^0 &+ (59\langle 1\rangle + 1\langle u\rangle)t^1 + 378\langle 1\rangle t^2 \\ &+ 2400\langle 1\rangle t^3 + 16566\langle 1\rangle t^4 + (117179\langle 1\rangle + 1\langle u\rangle)t^5 + \cdots\end{aligned}$$

9. Motivic measures

For k a field, we denote by $K_0(\mathrm{Var}_k)$ the *modified Grothendieck ring of varieties* over k, defined to be the quotient of the free abelian group on classes of algebraic varieties over k by the following relations:

$$X - Y - U \tag{25}$$

for every variety X over k and every closed subscheme Y of X with open complement U, and

$$X - Y \tag{26}$$

for all varieties X, Y over k such that there exists a radicial surjective morphism $f : X \to Y$. Recall that a morphism is said to be radicial surjective if it is bijective and if it induces purely inseparable extensions of residue fields.

For X a quasi-projective variety over a field k, Kapranov's zeta function [Kap00] is defined to be the power series with coefficients in the (modified) Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$ given by

$$Z_X^{\mathrm{Kap}}(t) = \sum_{n \geq 0} [\mathrm{Sym}^n(X)] t^n,$$

where $\mathrm{Sym}^n(X)$ is the n-th symmetric power of X. When $k = \mathbb{F}_q$, it specializes to $\zeta_X(t)$ via the counting measure $\#_{\mathbb{F}_q}$. It is therefore natural to ask whether one could also recover our enriched zeta function from Z_X^{Kap}. We do not see an immediate way of doing this. The natural candidate would be to apply the $\mathbb{A}^1$-categorical trace $\mathrm{Tr}(\varphi)$ where φ denotes the Frobenius

$$\mathrm{Tr}(\varphi) : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathrm{GW}(\mathbb{F}_q)$$

(See Proposition 9.1 to see that this is a motivic measure.) However, this gives the classical zeta function, with all coefficients in $\mathbb{Z} \subset \mathrm{GW}(\mathbb{F}_q)$ by [Hoy15, Example 1.6]. On the other hand, this motivates the question of determining whether our enriched trace and enriched zeta functions define *motivic measures* in some appropriate sense, and the aim of this section is to answer this question in full.

In this section, the endomorphism φ will be the Frobenius.

9.1. The $\mathbb{A}^1$-trace as a motivic measure.

Recall that for every $m \geq 1$, there is a motivic measure

$$\#_{\mathbb{F}_{q^m}} : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z},$$

called the counting measure, given by sending the class $[X]$ of a variety X over $\mathbb{F}_q$ to its point count $|X(\mathbb{F}_{q^m})|$. For every $m \geq 1$, define $N_m(X) = \mathrm{Tr}(\varphi^m) \in GW(\mathbb{F}_q)$ to be the $\mathbb{A}^1$-trace of the Frobenius endomorphism on X.

Proposition 9.1. *The assignment $X \mapsto N_m(X)$ induces a motivic measure*

$$N_m : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathrm{GW}(\mathbb{F}_q)$$

enriching $\#_{\mathbb{F}_q^m}$, in the sense that we recover $\#_{\mathbb{F}_q^m}$ by taking ranks.

Proof. We first show that N_m is well defined. For this, note that through the group isomorphism $\mathrm{GW}(\mathbb{F}_q) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, we have $N_m(X) = (|X(\mathbb{F}_q)|, \mathrm{disc}(N_m(X)))$, with $\mathrm{disc}(N_m(X))$ given by formula (24), and therefore it passes to the quotient with respect to both the cut-and-paste relations (25) and the relations (26).

To prove multiplicativity, note that the Frobenius on $X \times Y$ is given by the product $\varphi_X \times \varphi_Y$. The trace is multiplicative with respect to smash product [PS14, Corollary 5.9], giving the equality

$$N_m(X \times Y) = N_m(X) N_m(Y).$$

We may thus conclude that N_m defines a motivic measure. □

9.2. **The enriched zeta function as a motivic measure.** Associating to a variety X over $\mathbb{F}_q$ its zeta function $\zeta_X(t)$ induces a motivic measure

$$\zeta : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathcal{R}_1$$

where $\mathcal{R}_1 = \{f \in \mathbb{C}(t),\ f(0) = 1\} \subset 1 + t\mathbb{C}[[t]]$ is equipped with the Witt ring structure (see [Ram15, Theorem 2.1]). The logarithmic derivative

$$\mathrm{dlog} : 1 + t\mathbb{C}[[t]] \to \mathbb{C}[[t]]$$

sends the Witt ring structure to the ring structure where addition is addition of power series, and multiplication is coefficient-wise multiplication. In particular, composing it with ζ, we get a motivic measure

$$\mathrm{dlog}\,\zeta : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{C}^{\mathbb{N}}.$$

Proposition 9.2. *The assignment* $X \mapsto \mathrm{dlog}\,\zeta^{\mathbb{A}^1}_{X,\varphi}(t)$ *defines a motivic measure*

$$\mathrm{dlog}\,\zeta^{\mathbb{A}^1} : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathrm{GW}(k)[[t]]$$

lifting $\mathrm{dlog}\,\zeta$*.*

Proof. Follows from Proposition 9.1. □

Remark 9.3. The Kapranov zeta function $Z^{\mathrm{Kap}}_X(t)$ of any curve X over $\mathbb{F}_q$ is a rational function. On the other hand, the Kapranov zeta function is usually not rational for varieties of higher dimensions [LL03]. There is reason to believe that Morel and Sawant's conjectures [Mor22] on $\mathbb{A}^1$-cellular homology for general smooth projective varieties would have consequences for rationality of the $\mathbb{A}^1$-logarithmic zeta function analogous to Theorem 6.1.

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