EXPLICIT MINIMAL EMBEDDED RESOLUTIONS OF DIVISORS ON MODELS OF THE PROJECTIVE LINE

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Abstract. Let K be a discretely valued field with ring of integers $\mathcal{O}_K$ with perfect residue field. Let $K(x)$ be the rational function field in one variable. Let $\mathbb{P}^1_{\mathcal{O}_K}$ be the standard smooth model of $\mathbb{P}^1_K$ with coordinate x on irreducible special fiber. Let $f(x) \in \mathcal{O}_K[x]$ be a monic irreducible polynomial with corresponding divisor of zeroes $\operatorname{div}_0(f)$ on $\mathbb{P}^1_{\mathcal{O}_K}$. We give an explicit description of the minimal embedded resolution $\mathcal{Y}$ of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \operatorname{div}_0(f))$ by using Mac Lane's theory to write down the discrete valuations on $K(x)$ corresponding to the irreducible components of the special fiber of $\mathcal{Y}$.

1. Introduction

Let K be a discretely valued field with ring of integers $\mathcal{O}_K$ with perfect residue field. Let $K(x)$ be the rational function field in one variable. Let $\mathbb{P}^1_{\mathcal{O}_K}$ be the standard smooth model of $\mathbb{P}^1_K$ with coordinate x on irreducible special fiber. Let $f(x) \in \mathcal{O}_K[x]$ be a monic irreducible polynomial with corresponding divisor of zeroes $\operatorname{div}_0(f)$ on $\mathbb{P}^1_{\mathcal{O}_K}$. A minimal embedded resolution of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \operatorname{div}_0(f))$ is a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ with a birational morphism $\pi\colon \mathcal{Y} \to \mathbb{P}^1_{\mathcal{O}_K}$ such that the strict transform of $\operatorname{div}_0(f)$ is regular, and such that any other modification $\pi'\colon \mathcal{Y}' \to \mathbb{P}^1_{\mathcal{O}_K}$ with $\mathcal{Y}'$ regular and the strict transform of $\operatorname{div}_0(f)$ regular factors uniquely as $\mathcal{Y}' \to \mathcal{Y} \xrightarrow{\pi} \mathbb{P}^1_{\mathcal{O}_K}$. The main result of this paper is the following theorem (See Theorem 5.17 for a more precise statement, with notation as defined in Notation 3.9.)

Theorem 1.1. *Let $f \in \mathcal{O}_K[x]$ be a monic irreducible polynomial. There is an explicit description of the minimal embedded resolution $\mathcal{Y}$ of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \operatorname{div}_0(f))$ when $\deg(f) \geq 2$[1]. More specifically, we write down the discrete valuations on $K(x)$ corresponding to the irreducible components of the special fiber of $\mathcal{Y}$.*

Remark 1.2. Regular resolutions satisfy étale descent. That is, if L/K is an unramified field extension and $f \in \mathcal{O}_K[x]$ is a monic irreducible polynomial, then $\mathcal{Y}$ is an embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \operatorname{div}_0(f))$ if and only if $\mathcal{Y} \times_{\mathcal{O}_K} \mathcal{O}_L$ is an embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_L}, \operatorname{div}_0(f))$, in which case we have $\mathcal{Y} \cong (\mathcal{Y} \times_{\mathcal{O}_K} \mathcal{O}_L)/\operatorname{Gal}(L/K)$. Thus, in proving Theorem 1.1, we may assume that the residue field k of K is algebraically closed. We will indeed assume this throughout the paper.

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[1] When $\deg(f) = 1$, the divisor $\operatorname{div}_0(f)$ is already regular on the standard model $\mathbb{P}^1_{\mathcal{O}_K}$.

Remark 1.3. If $f = \pi_K f_1 \cdots f_r \in \mathcal{O}_K[x]$ is an irreducible factorization, let $\mathcal{Y}_i$ be the minimal embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f_i))$, and let $\mathcal{Y}'$ be the minimal normal model of $\mathbb{P}^1_K$ dominating all $\mathcal{Y}_i$. Then $\mathcal{Y}'$ is regular (see, e.g., [OS19b, Lemma 5.3]), and the minimal embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$ is the minimal blowup $\mathcal{Y} \to \mathcal{Y}'$ separating the strict transforms of $\mathrm{div}_0(\pi_K)$ and the $\mathrm{div}_0(f_i)$. Thus, neither the irreducibility nor the monicity of f is a serious condition, but our result can be stated much more cleanly when they are in place.

It is well-known that an algorithm for strong embedded resolution of singularities in dimension $n-1$ gives rise to an algorithm for resolution of singularities in dimension n. The motivation for the current paper is to explicitly understand regular models of cyclic covers of $\mathbb{P}^1_K$ by explicitly constructing embedded resolutions of pairs $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$ first. The eventual goal of these constructions is to give an upper bound on the number of components in such a resolution (See [OS19b] for an application to proving conductor-discriminant inequalities for degree 2 covers of $\mathbb{P}^1_K$, and forthcoming work of the authors for higher degree cyclic covers). We do so by capitalizing on the recent revival in [Rüt14, OW18] of explicit descriptions of normal and regular models of $\mathbb{P}^1_K$, using descriptions of valuations of $K(x)$ (now called "Mac Lane valuations") going back to Mac Lane [Mac36].

In [Rüt14, Proposition 3.4], Rüth shows that normal models of $\mathbb{P}^1_K$ are in bijection with non-empty finite collections of discrete valuations on $K(x)$ (extending the given valuation on K) whose residue fields have transcendence degree 1 over the residue field of K. Over algebraically closed fields, it is known that analogous valuations with value group $\mathbb{Q}$ on the rational function field can be constructed from supremum norms on non-archimedean disks. Over non-algebraically closed discretely valued fields, Rüth ([Rüt14, Proposition 4.56], restated in Proposition 2.7) shows that there is a similar description of valuations in terms of "diskoids", which are Galois stable collections of non-archimedean disks defined over the algebraic closure. In fact, he shows that these diskoids can be explicitly described by giving a certain sequence of polynomials φ_i in $K[x]$ of increasing degree ("centers" of a nested sequence of diskoids) and a corresponding sequence of rational numbers λ_i ("radii" of the diskoids) – such a description goes back to Mac Lane [Mac36] from 1936. These φ_i can be thought of as successive lower degree approximations to the roots of a polynomial $f \in \mathcal{O}_K[x]$, and each rational number λ_i is simply $\nu_K(\varphi_i(\alpha))$ for any root α of f (Corollary 2.11). Using successive φ_i-adic expansions, one can easily compute the valuation of any given polynomial from this description, by a procedure analogous to the computation of the Gauss valuation. (See the discussion surrounding Equation 2.1.) Mac Lane valuations have been implemented in Sage in [Rüt]. In [OW18, Theorem 7.8] (restated here in Proposition 3.10), the authors describe the minimal regular resolution of a model of $\mathbb{P}^1_K$ with irreducible special fiber corresponding to a valuation v, using the same polynomials φ_i that show up in the description of v, and natural Farey paths between successive λ_i.

In our setting, to each monic irreducible polynomial $f \in \mathcal{O}_K[x]$, there is a canonical diskoid centered about the roots of f giving rise to a valuation v_f (Section 4.1). By Rüth's correspondence, this valuation v_f corresponds to a normal model of $\mathbb{P}^1_K$ with irreducible special fiber, which we will call the v_f-model. By v_f-component, we mean the strict transform of the special fiber of the v_f-model in any model that dominates it. Concurrent to our work in [OS19a] (an earlier version of [OS19b]), in [KW20, Theorem 3.16], the authors also noted that $\mathrm{div}_0(f)$ is a normal crossings divisor on the minimal regular resolution $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ of the v_f-model $\mathcal{Y}_{v_f}$, which implies that the minimal normal model $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$ dominating $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ and $\mathbb{P}^1_{\mathcal{O}_K}$ is an embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$. However, $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$ is never the *minimal* embedded resolution of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$. In fact, for the applications to regular models of hyperelliptic curves, we are sometimes forced to work with (regular) contractions of $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$ where the strict transform of $\mathrm{div}_0(f)$ is also regular. Determining whether the horizontal part of $\mathrm{div}_0(f)$ remains regular on these contractions can be challenging because it might specialize to a node.

The two main insights of this paper are the following. First, using the machinery of Mac Lane valuations, it is possible to explicitly modify f to write down a rational function g that cuts out the unique irreducible *horizontal* divisor in $\mathrm{div}_0(f)$ on natural contractions of $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$. Note that checking regularity of $\mathrm{div}_0(g)$ at its unique closed point y is equivalent to checking whether g is in the square of the maximal ideal at y. This is hard to check directly since this local ring is 2-dimensional. The second main insight is to use the φ_n-adic expansion of f[2] to write down an analogous explicit decomposition $g = \sum_i g_i$. The terms g_i in this decomposition vanish along *vertical* components through the closed point y (a computation back in a 1-dimensional local ring), even though g itself does not, and we can exploit the orders of vanishing to determine when g is in the square of the maximal ideal. It turns out that the Mac Lane descriptions of vertical components are tailor-made for computing orders of vanishing of functions along these components!

Our main theorem shows that quite often it is possible to contract entire tails in the dual graph of $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$ and in fact, the minimal embedded resolution we are after is the minimal regular resolution of one of two neighbouring components of the v_f-component in the dual graph of $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$. We do not see any way to deduce our main theorem directly from [KW20, Theorem 3.16].

1.1. **Outline of the paper.** In §2, we introduce Mac Lane valuations. As we have mentioned, a normal model of $\mathbb{P}^1_K$ corresponds to a finite set of Mac Lane valuations, one valuation for each irreducible component of the special fiber. Mac Lane valuations are also in one-to-one correspondence with *diskoids*, which are Galois orbits of rigid-analytic disks in $\mathbb{P}^1_{\overline{K}}$. We will use the diskoid perspective often, and it is introduced in in §2.2.

In §3, we prove several results about the correspondence between Mac Lane valuations and normal models of $\mathbb{P}^1_K$. For instance, if $\mathcal{Y}$ is a normal model of $\mathbb{P}^1_K$ with special fiber consisting of several irreducible components, each corresponding to a Mac Lane valuation, results in §3 can be used to determine which irreducible component a point of $\mathbb{P}^1_K$ specializes to. After this, we cite a result (Proposition 3.10) from [OW18] giving an explicit criterion for when a normal model of $\mathbb{P}^1_K$ is regular. More specifically, using that Mac Lane valuations correspond to normal models of $\mathbb{P}^1_K$ with irreducible special fiber, Proposition 3.10 takes a Mac Lane valuation as input and gives the minimal regular resolution of the corresponding normal model as output (as a finite set of Mac Lane valuations, of course)!

In §4, we first define the canonical valuation v_f associated to a polynomial f. The minimal embedded resolution of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$ is a certain contraction of $\mathcal{Y}_{v_f,0}^{\mathrm{reg}}$. So we are lead to an analysis of regularity of the strict transform of $\mathrm{div}_0(f)$, which we will henceforth call D, on natural contractions of $\mathcal{Y}'$. To this end, in Section 4 we first define three types of

[2]here φ_n is the last polynomial that shows up in the Mac Lane description of v_f

regular models of $\mathbb{P}^1_K$ that can arise as contractions of $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$. Viewing these contractions as a sequence of closed point blow-downs, a short argument shows that if we want the blow-down to stay regular and dominate $\mathbb{P}^1_{\mathcal{O}_K}$, there is a unique component that can be blown down at every stage (for instance, the v_f-component is the only -1-component that can be blown down in the model $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$ by the *minimality* of the construction of $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$). As we proceed through this natural sequence of blow-downs, we first go through a sequence of models we call "Type I" models. If D stays regular on all Type I regular blow-downs of $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$, we then move on to the "Type II" models. We continue contracting in this way, and after the Type II models, naturally comes the unique "Type III" model. (See Definition 4.4.)

The crux of the argument is to show that D is *not* regular on the unique Type III model (Proposition 5.13), and we use this to show that the minimal embedded resolution of D must be a special Type I or a Type II model (Corollary 5.14). We then show that if D is regular on a Type I or Type II model, then the model must include a component corresponding to one of two additional canonical valuations attached to the polynomial f, denoted v'_f, v''_f (Proposition 5.15) – these turn out to be neighbouring valuations to v_f in the dual graph of $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$. The technical lemmas needed for these regularity arguments use an analysis of valuations of individual terms in the φ-adic expansion of f along *vertical* components of these models (Lemma 5.7 for the unique Type III model, and Lemma 5.5 for Type I and Type II models). The Mac Lane machinery for describing these vertical components is perfectly equipped for carrying out such calculations. Finally, in Theorem 5.17, we show that the minimal embedded resolution of the pair $(\mathbb{P}^1_{\mathcal{O}_K}, \mathrm{div}_0(f))$ is the minimal regular model dominating $\mathbb{P}^1_{\mathcal{O}_K}$ and either the v'_f-model or the v''_f-model.

Notation and conventions

Throughout, K is a Henselian field with respect to a discrete valuation ν_K. We further assume that the residue field k of K is *algebraically closed*. We denote *fixed* separable and algebraic closures of K by $K^{\mathrm{sep}} \subseteq \overline{K}$. All algebraic extensions of K are assumed to live inside $\overline{K}$. This means that for any algebraic extension L/K, there is a preferred embedding $\iota_L \in \mathrm{Hom}_K(L, \overline{K})$, namely the inclusion. We fix a uniformizer π_K of ν_K and normalize ν_K so that $\nu_K(\pi_K) = 1$. Note that the valuation ν_K uniquely extends to a valuation on $\overline{K}$.

For an integral K-scheme or $\mathcal{O}_K$-scheme S, we denote the corresponding function field by $K(S)$. If $\mathcal{Y} \to \mathcal{O}_K$ is an arithmetic surface, an irreducible codimension 1 subscheme of $\mathcal{Y}$ is called *vertical* if it lies in a fiber of $\mathcal{Y} \to \mathcal{O}_K$, and *horizontal* otherwise. Let $f \in K(\mathcal{Y})$. We denote the divisor of zeroes of f by $\mathrm{div}_0(f)$. For any discrete valuation v, we denote the corresponding value group by Γ_v. If E is a regular codimension 1 point of $\mathcal{Y}$, we will denote the corresponding discrete valuation on the function field of $\mathcal{Y}$ by ν_E. If P is a closed point on $\mathcal{Y}$, we denote the corresponding local ring by $\mathcal{O}_{\mathcal{Y},P}$ and maximal ideal by $\mathfrak{m}_{\mathcal{Y},P}$.

Throughout this paper, we fix a system of homogeneous coordinates $\mathbb{P}^1_K = \mathrm{Proj}\, K[x_0, x_1]$, and $x := x_1/x_0$ and $\mathbb{P}^1_{\mathcal{O}_K} := \mathrm{Proj}\, \mathcal{O}_K[x_0, x_1]$.

All minimal polynomials are assumed to be monic. When we refer to the *denominator* of a rational number, we mean the positive denominator when the rational number is expressed as a reduced fraction.

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2. Mac Lane valuations

2.1. **Definitions and facts.** We recall the theory of inductive valuations, which was first developed by Mac Lane in [Mac36]. We also use the more recent [Rüt14] as a reference. Inductive valuations give us an explicit way to talk about normal models of $\mathbb{P}^1$.

Define a *geometric valuation* of $K(x)$ to be a discrete valuation that restricts to ν_K on K and whose residue field is a finitely generated extension of k with transcendence degree 1. We place a partial order $\preceq$ on valuations by defining $v \preceq w$ if $v(f) \leq w(f)$ for all $f \in K[x]$. Let v_0 be the *Gauss valuation* on $K(x)$. This is defined on $K[x]$ by $v_0(a_0 + a_1 x + \cdots a_n x^n) = \min_{0 \leq i \leq n} \nu_K(a_i)$, and then extended to $K(x)$.

We consider geometric valuations v such that $v \succeq v_0$. By the triangle inequality, these are precisely those geometric valuations for which $v(x) \geq 0$. This entails no loss of generality, since x can always be replaced by x^{-1}. We would like an explicit formula for describing geometric valuations, similar to the formula above for the Gauss valuation, and this is achieved by the so-called *inductive valuations* or *Mac Lane valuations*. Observe that the Gauss valuation is described using the x-adic expansion of a polynomial. The idea of a Mac Lane valuation is to "declare" certain polynomials φ_i to have higher valuation than expected, and then to compute the valuation recursively using φ_i-adic expansions.

More specifically, if v is a geometric valuation such that $v \succeq v_0$, the concept of a *key polynomial* over v is defined in [Mac36, Definition 4.1] (or [Rüt14, Definition 4.7]). Key polynomials are monic polynomials in $\mathcal{O}_K[x]$ — we do not give a definition, which would require more terminology than we need to develop, but see Lemma 2.3 below for the most useful properties. If $\varphi \in \mathcal{O}_K[x]$ is a key polynomial over v, then for $\lambda > v(\varphi)$, we define an *augmented valuation* $v' = [v, v'(\varphi) = \lambda]$ on $K[x]$ by

$$v'(a_0 + a_1\varphi + \cdots + a_r\varphi^r) = \min_{0 \leq i \leq r} v(a_i) + i\lambda \tag{2.1}$$

whenever the $a_i \in K[x]$ are polynomials with degree less than $\deg(\varphi)$. We should think of this as a "base φ expansion", and of $v'(f)$ as being the minimum valuation of a term in the base φ expansion of f when the valuation of φ is declared to be λ. By [Mac36, Theorems 4.2, 5.1] (see also [Rüt14, Lemmas 4.11, 4.17]), v' is in fact a discrete valuation. In fact, the key polynomials are more or less the polynomials φ for which the construction above yields a discrete valuation for $\lambda > v(\varphi)$. The valuation v' extends to $K(x)$.

We extend this notation to write Mac Lane valuations in the following form:

$$[v_0, v_1(\varphi_1(x)) = \lambda_1, \ldots, v_n(\varphi_n(x)) = \lambda_n].$$

Here each $\varphi_i(x) \in \mathcal{O}_K[x]$ is a key polynomial over v_{i-1}, we have that $\deg(\varphi_{i-1}(x)) \mid \deg(\varphi_i(x))$, and each λ_i satisfies $\lambda_i > v_{i-1}(\varphi_i(x))$. By abuse of notation, we refer to such a valuation as v_n (if we have not given it another name), and we identify v_i with $[v_0, v_1(\varphi_1(x)) = \lambda_1, \ldots, v_i(\varphi_i(x)) = \lambda_i]$ for each $i \leq n$. The valuation v_i is called a *truncation* of v_n. One sees without much difficulty that $v_n(\varphi_i) = \lambda_i$ for all i between 1 and n.

It turns out that the set of Mac Lane valuations on $K(x)$ exactly coincides with the set of geometric valuations v with $v \succeq v_0$ ([FGMN15, Corollary 7.4] and [Mac36, Theorem 8.1], or [Rüt14, Theorem 4.31]). Furthermore, every Mac Lane valuation is equal to one where the degrees of the φ_i are strictly increasing ([Mac36, Lemma 15.1] or [Rüt14, Remark 4.16]), so we may and do assume this to be the case for the rest of the paper. This has the consequence that the number n is well-defined. We call n the *inductive valuation length* of v. In fact, by [Mac36, Lemma 15.3] (or [Rüt14, Lemma 4.33]), the degrees of the φ_i and the values of the λ_i are invariants of v, once we require that they be strictly increasing. If f is a key polynomial over $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ and either $\deg(f) > \deg(\varphi_n)$ or $v = v_0$, we call f a *proper key polynomial over* v. By our convention, each φ_i is a proper key polynomial over v_{i-1}.

In general, if v and w are two Mac Lane valuations such that the value group Γ_w contains the value group Γ_v, we write $e(w/v)$ for the ramification index $[\Gamma_w : \Gamma_v]$.

Remark 2.2. Observe that if $[v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ is a Mac Lane valuation, where each $\lambda_i = b_i/c_i$ in lowest terms, then the ramification index $e(v_n/v_0)$ equals $\operatorname{lcm}(c_1, \ldots, c_n)$. Consequently, $e(v_i/v_j) = \operatorname{lcm}(c_1, \ldots, c_i)/\operatorname{lcm}(c_1, \ldots, c_j)$ for $i \geq j$.

We collect some basic results on Mac Lane valuations and key polynomials that will be used repeatedly.

Lemma 2.3. *Suppose f is a proper key polynomial over $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$.*

- (i) *If $n = 0$, then f is linear. If $n \geq 1$, then φ_1 is linear. Every monic linear polynomial in $\mathcal{O}_K[x]$ is a key polynomial over v_0.*
- (ii) *If $n \geq 1$, and $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ is the φ_n-adic expansion of f, then $v_n(a_0) = v_n(\varphi_n^e) = e\lambda_n$, and $v_n(a_i\varphi_n^i) \geq e\lambda_n$ for all $i \in \{1, \ldots, e-1\}$. In particular, $v_n(f) = e\lambda_n$.*
- (iii) *If $n \geq 1$, then $\deg(f)/\deg(\varphi_n) = e(v_n/v_{n-1})$.*

Proof. Part (i) follows from [OW18, Remark 5.2(i)] for $n = 0$, and then for general $n \geq 1$ by applying the $n = 0$ case to φ_1 and v_0. Part (ii) follows from [Mac36, Theorem 9.4] (or [Rüt14, Lemma 4.19(ii), (iii)]). Part (iii) follows from [Mac36, Theorem 12.1] (one can also use the second equation of [Rüt14, Lemma 4.30], where $\mathbb{F}_m = \mathbb{F}_{m-1} = k$, but note that [Rüt14, Lemma 4.30] is incorrect as stated — the expression $e(v_m/v_{m-1})$ should be replaced by $e(v_{m-1}/v_{m-2})$). □

Corollary 2.4. *Let $v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation of inductive valuation length $n \geq 1$. Write $\lambda_i = b_i/c_i$ in lowest terms for all i. Let $N_n = \operatorname{lcm}_{i<n} c_i$ if $n > 1$, and let $N_n = 1$ if $n = 1$. Then $N_n = e(v_{n-1}/v_0) = \deg(\varphi_n)$, and thus $\Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z} = (1/\deg(\varphi_n))\mathbb{Z}$.*

Proof. That $\deg(\varphi_1) = 1$ is Lemma 2.3(i), which proves the corollary when $n = 1$. By Remark 2.2, $e(v_{j+1}/v_j) \operatorname{lcm}(c_1, \ldots, c_j) = \operatorname{lcm}(c_1, \ldots, c_{j+1})$. The rest of the corollary follows from Lemma 2.3(iii) and induction. □

Lemma 2.5. *Let $[v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ be a valuation over which f is a proper key polynomial. Then for $1 \leq i \leq n$, we have $\lambda_i \notin \Gamma_{v_{i-1}} = (1/N_i)\mathbb{Z}$.*

Proof. If $\lambda_i \in \Gamma_{v_{i-1}}$, then $e(v_i/v_{i-1}) = 1$. If $i = n$, applying Lemma 2.3(iii) to v_n, contradicts the fact that $\deg(f) > \deg(\varphi_n)$. For $i < n$, applying Lemma 2.3(iii) to v_i contradicts the fact that $\deg(\varphi_{i+1}) > \deg(\varphi_i)$. □

Example 2.6. If $K = \mathrm{Frac}(W(\overline{\mathbb{F}}_3))$, then the polynomial $f(x) = x^3 - 9$ is a proper key polynomial over $[v_0,\, v_1(x) = 2/3]$. In accordance with Lemma 2.3(ii), we have $v_1(f) = v_1(9) = v_1(x^3) = 3 \cdot 2/3 = 2$. If we extend v_1 to a valuation $[v_0,\, v_1(x) = 2/3,\, v_2(f(x)) = \lambda_2]$ with $\lambda_2 > 2$, then the valuation v_2 notices "cancellation" in $x^3 - 9$ that v_1 does not.

2.2. **Mac Lane valuations and diskoids.** Given $\varphi \in \mathcal{O}_K[x]$ monic, irreducible and $\lambda \in \mathbb{Q}_{\geq 0}$, we define the *diskoid* $D(\varphi, \lambda)$ with "center" φ and radius λ to be $D(\varphi, \lambda) := \{\alpha \in \overline{K} \mid \nu_K(\varphi(\alpha)) \geq \lambda\}$ (we only treat diskoids with *non-negative, finite* radius in the sense of [Rüt14, Definition 4.40]). By [Rüt14, Lemma 4.43], a diskoid is a union of a disk with all of its $\mathrm{Gal}(\overline{K}/K)$-conjugates. Such a diskoid is said to be *defined* over K, since $\varphi \in \mathcal{O}_K[x]$. Notice that the *larger* λ is, the *smaller* the diskoid is. We now state the fundamental correspondence between Mac Lane valuations and diskoids.

Proposition 2.7 (cf. [Rüt14, Theorem 4.56], see also [OW18, Proposition 5.4])**.** *There is a bijection from the set of diskoids to the set of Mac Lane valuations that sends a diskoid D to the valuation v_D defined by $v_D(f) = \inf_{\alpha \in D} \nu_K(f(\alpha))$. The inverse sends a Mac Lane valuation $v = [v_0,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$ to the diskoid D_v defined by $D_v = D(\varphi_n, \lambda_n)$. Alternatively,*

$$D_v = \{\alpha \in \overline{K} \ \mid \ \nu_K(f(\alpha)) \geq v(f) \ \forall f \in K[x]\},$$

is a presentation of D_v independent of the description of v as a Mac Lane valuation.

Lastly, if D and D' are diskoids, then $D \subseteq D'$ if and only if $v_D \succeq v_{D'}$. If v and v' are Mac Lane valuations, then $v \succeq v'$ if and only if $D_v \subseteq D_{v'}$.

The following proposition is crucial for our method.

Proposition 2.8. *Let $\alpha \in \mathcal{O}_{\overline{K}}$, and let $f \in K[x]$ be the minimal polynomial for α. Then there exists a unique Mac Lane valuation $v_f = [v_0,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$ over which f is a proper key polynomial.*

Proof. Consider the unique valuation v_L on $L := K[x]/(f)$ extending v_K. This lifts to a discrete *pseudovaluation* on $K[x]$ in the language of [Rüt14, §4.6] (a valuation which can take the value ∞ on an ideal, in this case (f)). By [Rüt14, Corollary 4.67], it can be written as a so-called "infinite inductive valuation" $[v_0,\, \ldots,\, v_n(\varphi_n) = \lambda_n, v_{n+1}(f) = \infty]$, with f a proper key polynomial over $v_f := [v_0,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$. This shows the existence of v_f. If f is a proper key polynomial over some other valuation v, then for sufficiently large λ, one can construct inductive valuations $v'_f = [v_f,\, v'_f(f) = \lambda]$ and $v' = [v,\, v'(f) = \lambda]$. By Proposition 2.7, these inductive valuations correspond to the same diskoid, and are thus the same. Applying the "only if" direction of [Rüt14, Theorem 4.33] (or [Mac36, Theorem 15.3]) to v'_f and v', and then the "if" direction of the same theorem to v_f and v shows that $v_f = v$. □

To close out §2.2, we prove several results linking Mac Lane valuations evaluated at a polynomial to the valuation of that polynomial at a particular point.

Definition 2.9 ([Rüt14, Definition 4.4, Lemma 4.24])**.** If $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ is a Mac Lane valuation and $f \in K[x]$, then a *v-reciprocal* of f is a polynomial $f' \in K[x]$ such that $v(ff' - 1) > 0$ and $v(f') = v_{n-1}(f') = -v(f)$.

By [Mac36, Lemma 9.1] (or [Rüt14, Lemma 4.24]), any $f \in K[x]$ with $v(f) = v_{n-1}(f)$ has a v-reciprocal. In this case, it is clear from Definition 2.9 that f and f' being v-reciprocals is a symmetric relation.

Proposition 2.10. *Suppose* $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ *is a Mac Lane valuation,* $\alpha \in D(\varphi_n, \lambda_n)$*, and* $g \in K[x]$ *such that* $v(g) = v_{n-1}(g)$*. Then* $\nu_K(g(\alpha)) = v(g)$*.*

Proof. Let $D := D(\varphi_n, \lambda_n)$ be the diskoid corresponding to v and let $D' := D(g, \nu_K(g(\alpha)))$ with corresponding valuation v'. These two diskoids share the common element α. By [Rüt14, Lemma 4.44], either $D \subseteq D'$ or $D' \subseteq D$, and then Proposition 2.7 shows that either $v' \preceq v$ or $v \preceq v'$.

Since $\alpha \in D$, by Proposition 2.7 we have $\nu_K(g(\alpha)) \geq v(g)$. Suppose $\nu_K(g(\alpha)) > v(g)$. Since $v'(g) = \nu_K(g(\alpha))$ by definition, we have $v(g) < v'(g)$. Since either $v' \preceq v$ or $v \preceq v'$, it follows that $v \preceq v'$. Let $g' \in K[x]$ be a v-reciprocal of g, i.e., $gg' = 1 + h$ with $v(h) > 0$ (g' exists because $v(g) = v_{n-1}(g)$). Since $v \preceq v'$, we have $0 < v(h) \leq v'(h)$. In particular, $v'(gg') = v(gg') = 0$, so $v'(g') = -v'(g) < -v(g) = v(g')$. But this contradicts $v \preceq v'$. ☐

Corollary 2.11. *If* f *is a key polynomial over* $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$ *with root* $\alpha \in \overline{K}$*, then* $\nu_K(g(\alpha)) = v(g)$ *for all* $g \in \mathcal{O}_K[x]$ *of degree less than* $\deg(f)$*. In particular,* $\nu_K(\varphi_i(\alpha)) = \lambda_i$ *for all* $1 \leq i \leq n$*.*

Proof. Consider a Mac Lane valuation $w_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_n(\varphi_n) = \lambda_n,\, v_{n+1}(f) = \lambda_{n+1}]$, with λ_{n+1} large. Then $v_{n+1}(g) = v_n(g)$ and $\alpha \in D(f, \lambda_{n+1})$, so the corollary follows from Proposition 2.10. ☐

3. Mac Lane valuations, normal models and regular resolutions

In §3.1, we prove results on the specialization of horizontal divisors, expressed in terms of Mac Lane valuations. In §3.2 we recall a result from [OW18], giving a criterion in terms of Mac Lane valuations for when a model of $\mathbb{P}^1_K$ is regular. Lastly, in §3.3, we discuss valuations that are in a geometric sense "nearby" to a given Mac Lane valuation in a regular model of $\mathbb{P}^1_K$. These valuations will play a crucial role throughout the rest of the paper.

A *normal model* of $\mathbb{P}^1_K$ is a flat, normal, proper $\mathcal{O}_K$-curve with generic fiber isomorphic to $\mathbb{P}^1_K$. By [Rüt14, Corollary 3.18], normal models $\mathcal{Y}$ of $\mathbb{P}^1_K$ are in one-to-one correspondence with non-empty finite collections of geometric valuations, by sending $\mathcal{Y}$ to the collection of geometric valuations corresponding to the local rings at the generic points of the irreducible components of the special fiber of $\mathcal{Y}$. Via this correspondence, the multiplicity of an irreducible component of the special fiber of a normal model $\mathcal{Y}$ of $\mathbb{P}^1_K$ corresponding to a Mac Lane valuation v equals $e(v/v_0)$.

We say that a normal model of $\mathbb{P}^1_K$ *includes* a Mac Lane valuation v if a component of the special fiber corresponds to v. If $\mathcal{Y}$ includes v, we call the corresponding irreducible component of its special fiber the *v-component* of the special fiber of $\mathcal{Y}$ (or simply the v-component of $\mathcal{Y}$, even though it is not an irreducible component of $\mathcal{Y}$). If S is a finite set of Mac Lane valuations, then the *S-model of* $\mathbb{P}^1_K$ is the normal model including exactly the valuations in S. If $S = \{v\}$, we simply say the v-model instead of the $\{v\}$-model. Recall that we fixed a coordinate x on $\mathbb{P}^1_K$, that is, a rational function x on $\mathbb{P}^1_K$ such that $K(\mathbb{P}^1_K) = K(x)$.

3.1. Specialization of horizontal divisors.

Each $\alpha \in \overline{K} \cup \{\infty\}$ corresponds to a point of $\mathbb{P}^1(\overline{K})$ given by $x = \alpha$, which lies over a unique closed point of $\mathbb{P}^1_K$. If $\mathcal{Y}$ is a normal model of $\mathbb{P}^1_K$, the closure of this point in $\mathcal{Y}$ is a subscheme that we call D_α; note that D_α is a horizontal divisor (the model will be clear from context, so we omit it to lighten the notation).

If v is a Mac Lane valuation, then the reduced special fiber of the v-model of $\mathbb{P}^1_K$ is isomorphic to $\mathbb{P}^1_k$ (see, e.g., [OW18, Lemma 7.1]). It will be useful to have an explicit coordinate on this special fiber (that is, a rational function y such that the function field of the special fiber is $k(y)$).

Lemma 3.1. *Let $v = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation, and let $e = e(v_n/v_{n-1})$. There exists a monomial t in $\varphi_1, \dots, \varphi_{n-1}$ such that $v(t\varphi_n^e) = 0$, and for any such t, the restriction of $t\varphi_n^e$ to the reduced special fiber of the v-model of $\mathbb{P}^1_K$ is a coordinate on the v-component that vanishes at the specialization of $\varphi_n = 0$.*

Proof. Let $\mathcal{O} \subseteq K[x]$ be the subring of elements f such that $v(f) \geq 0$, and let $\mathcal{O}^+$ be the ideal of elements g where $v(g) > 0$. Let $e = e(v_n/v_{n-1})$. By [Mac36, Theorem 12.1] (or [Rüt14, Lemma 4.29] and the discussion before that lemma), $\mathcal{O}/\mathcal{O}^+ \cong k[y]$, where y is the image of $t\varphi_n^e$ in $\mathcal{O}/\mathcal{O}^+$, for any $t \in K[x]$ with $v(t\varphi_n^e) = 0$ and $v(t) = v_{n-1}(t)$ (in the notation of [Rüt14], the example used is $t = (S')^\ell$). Since $v(\varphi_n^e) \in \Gamma_{v_{n-1}}$, we can take t to be a monomial in $\varphi_1, \dots, \varphi_{n-1}$. Since Spec $\mathcal{O}$ is an affine open of the v-model with reduced special fiber Spec $\mathcal{O}/\mathcal{O}^+ \cong$ Spec $k[y] \cong \mathbb{A}^1_k \subseteq \mathbb{P}^1_k$, we have that y is a coordinate on the reduced special fiber of the v-model of $\mathbb{P}^1_K$. $\square$

Proposition 3.2. *Let $v = [v_0, v_1(\varphi_1) = \lambda_1, \dots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation and let $\mathcal{Y}$ be the v-model of $\mathbb{P}^1_K$. As α ranges over $\overline{K}$, all D_α with $\nu_K(\varphi_n(\alpha)) > \lambda_n$ meet on the special fiber, all D_α with $\nu_K(\varphi_n(\alpha)) < \lambda_n$ meet at a different point on the special fiber, and no D_α with $\nu_K(\varphi_n(\alpha)) \neq \lambda_n$ meets any D_β with $\nu_K(\varphi_n(\beta)) = \lambda_n$.*

Proof. Let $\mathcal{Y}$ be the v-model of $\mathbb{P}^1_K$. Using the coordinate $y := t\varphi_n^e$ from Lemma 3.1 on the reduced special fiber of $\mathcal{Y}$, we will show that all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) < \lambda_n$ specialize to $y = \infty$, all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) > \lambda_n$ specialize to $y = 0$ and all $\alpha \in \overline{K}$ with $\nu_K(\varphi_n(\alpha)) = \lambda_n$ specialize to some point $y = a$ with $a \notin \{0, \infty\}$. We now work out the details.

Let $\mathcal{O} \subseteq K[x]$ be the subring of elements f such that $v(f) \geq 0$, and let $\mathcal{O}^+$ be the ideal of elements g where $v(g) > 0$. Suppose $\alpha \in D(\varphi_n, \lambda_n)$. Proposition 2.7 shows that $\nu_K(g(\alpha)) > 0$ for $g \in \mathcal{O}^+$, thus evaluating y at α gives a well-defined element of k. Furthermore, $y = y(\alpha)$ is precisely the point where D_α meets the special fiber of $\mathcal{Y}$. We now compute:

$$\begin{aligned} y(\alpha) = 0 &\Leftrightarrow \nu_K(t(\alpha)\varphi_n(\alpha)^e) > 0 \\ &\Leftrightarrow \nu_K(t(\alpha)\varphi_n(\alpha)^e) > v(t\varphi_n^e) \\ &\Leftrightarrow \nu_K(\varphi_n(\alpha)) > \lambda_n \qquad\qquad (\because\ \nu_K(t(\alpha)) = v(t)). \end{aligned}$$

This shows that all D_α for which $\nu_K(\varphi_n(\alpha)) > \lambda_n$ intersect on the special fiber at the point $y = 0$, but none of them intersect any D_β for which $\nu_K(\varphi_n(\beta)) = \lambda_n$. All such D_β intersect the reduced special fiber $\mathbb{A}^1_k \cong$ Spec $k[y]$ of Spec $\mathcal{O}$ at some point where $y \neq 0$.

Now let $\alpha \notin D(\varphi_n, \lambda_n)$. We will show that $D_\alpha \cap (\text{Spec } \mathcal{O})_s$ is empty by contradiction. Suppose not. Let $P \in D_\alpha \cap (\text{Spec } \mathcal{O})_s$ be a closed point of Spec $\mathcal{O}$. We have a well-defined element $g(P) \in k$ for every $g \in \mathcal{O}$ coming from evaluating g at P. Since P is the closed point of $D_\alpha \cong$ Spec A with $A \subseteq \mathcal{O}_{K(\alpha)}$, it follows that $g(\alpha) \in \mathcal{O}_{K(\alpha)}$ and furthermore, $g(P) = g(\alpha)$

mod $\mathfrak{m}_{\mathcal{O}_{K(\alpha)}}$. We will now construct a $g \in \mathcal{O}$ with $\nu_K(g(\alpha)) < 0$, which is a contradiction. Let b be such that $bv(\varphi_n) \in \mathbb{Z}_{>0}$, and let $g := \varphi_n^b / \pi_K^{bv(\varphi_n)}$. Then $v(g) = 0$ so $g \in \mathcal{O}$, but

$$\nu_K(g(\alpha)) = b(\nu_K(\varphi_n(\alpha)) - v(\varphi_n)) < 0.$$

Thus D_α does not intersect the special fiber of Spec $\mathcal{O}$, so D_α specializes to a point of $\mathcal{Y}_s \setminus (\text{Spec } \mathcal{O})_s$, which is the "point at infinity" where $y = \infty$ on the reduced special fiber of $\mathcal{Y}$. This finishes the proof. ☐

Corollary 3.3. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation and let $\mathcal{Y}$ be a normal model of $\mathbb{P}^1_K$ including v. If $\alpha, \beta \in \overline{K}$ are such that $\nu_K(\varphi_n(\beta)) \leq \lambda_n \leq \nu_K(\varphi_n(\alpha))$ and $\nu_K(\varphi_n(\beta)) \neq \nu_K(\varphi_n(\alpha))$, then D_α and D_β do not meet on the special fiber of $\mathcal{Y}$.*

Proof. Immediate from Proposition 3.2. ☐

Corollary 3.4. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]$ and $v' = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots, v'_n(\varphi_n) = \lambda'_n]$ be Mac Lane valuations with $\lambda'_n < \lambda_n$. Let $\mathcal{Y}$ be a model of $\mathbb{P}^1_K$ including v and v' on which the v- and v'-components intersect, say at a point z. Then D_α meets z if and only if $\lambda'_n < \nu_K(\varphi_n(\alpha)) < \lambda_n$.*

Proof. We may assume $\mathcal{Y}$ is the $\{v, v'\}$-model $\mathbb{P}^1_K$. Let $\overline{Y}$ and $\overline{Y}'$ be the v and v'-components of $\mathcal{Y}$, respectively, so that $z = \overline{Y} \cap \overline{Y}'$. First suppose $\lambda'_n < \nu_K(\varphi_n(\alpha)) < \lambda_n$. If D_α meets a point of $\overline{Y} \setminus \overline{Y}'$, then by Proposition 3.2 applied to the blow down of $\overline{Y}' \subseteq \mathcal{Y}$ (i.e., the v-model of $\mathbb{P}^1_K$), all D_α outside of $D(\varphi_n, \lambda_n)$ intersect this point on $\overline{Y} \subseteq \mathcal{Y}$. So if we blow down $\overline{Y} \subseteq \mathcal{Y}$, then all D_α for $\alpha \notin D(\varphi_n, \lambda_n)$ specialize to the same point. Since we can find $\alpha_1, \alpha_2 \in \overline{K} \setminus D(\varphi_n, \lambda_n)$ with $\nu_K(\varphi_n(\alpha_1)) = \lambda'_n$ and $\lambda'_n < \nu_K(\varphi_n(\alpha_2)) < \lambda_n$, the previous line contradicts Proposition 3.2 applied to the v'-model of $\mathbb{P}^1_K$. The same argument applied to the blow down of $\overline{Y}$ (i.e, the v-model of $\mathbb{P}^1_K$) yields a contradiction if D_α intersects a point of $\overline{Y}' \setminus \overline{Y}$. So D_α meets the intersection point z of the two irreducible components of the special fiber.

Now, suppose $\nu_K(\varphi_n(\alpha)) \leq \lambda'_n$. Fix $\beta \in \overline{K}$ such that $\lambda'_n < \nu_K(\varphi_n(\beta)) < \lambda_n$. Corollary 3.3 shows that D_α and D_β do not meet on the v'-model of $\mathbb{P}^1_K$, and thus not on $\mathcal{Y}$ either. In particular, since D_β meets z by the previous paragraph, D_α does not. A similar proof works if $\nu_K(\varphi_n(\alpha)) \geq \lambda_n$ using the v-model instead of the v'-model. This completes the proof of the corollary. ☐

3.2. **Resolution of singularities on normal models of $\mathbb{P}^1$.** Let $\mathcal{Y}$ be a normal model of $\mathbb{P}^1_K$. A *minimal regular resolution of* $\mathcal{Y}$ is a (proper) regular model $\mathcal{Z}$ of $\mathbb{P}^1_K$ with a surjective, birational morphism $\pi : \mathcal{Z} \to \mathcal{Y}$ such that the special fiber of $\mathcal{Z}$ contains no -1-components ([CES03, Definition 2.2.1]). Such minimal regular resolutions exist and are unique, e.g., by [CES03, Theorem 2.2.2].

In the remainder of §3.2, we recall a fundamental result from [OW18], which expresses minimal regular resolutions of models of $\mathbb{P}^1_K$ with irreducible special fiber in terms of Mac Lane valuations.

3.2.1. *Shortest N-paths.* We start by recalling the notion of *shortest N-path*, introduced in [OW18].

Definition 3.5. Let N be a natural number, and let $a > a' \geq 0$ be rational numbers. An *N-path* from a to a' is a decreasing sequence $a = b_0/c_0 > b_1/c_1 > \cdots > b_r/c_r = a'$ of rational numbers in lowest terms such that

$$\frac{b_i}{c_i} - \frac{b_{i+1}}{c_{i+1}} = \frac{N}{\operatorname{lcm}(N, c_i)\operatorname{lcm}(N, c_{i+1})}$$

for $0 \leq i \leq r-1$. If, in addition, no proper subsequence of $b_0/c_0 > \cdots > b_r/c_r$ containing b_0/c_0 and b_r/c_r is an N-path, then the sequence is called the *shortest N-path* from a to a'.

Remark 3.6. By [OW18, Proposition A.14], the shortest N-path from a' to a exists and is unique.

Remark 3.7. Observe that two successive entries $b_i/c_i > b_{i+1}/c_{i+1}$ of a shortest 1-path satisfy $b_i/c_i - b_{i+1}/c_{i+1} = 1/(c_i c_{i+1})$.

Example 3.8. The sequence $1 > 1/2 > 2/5 > 3/8 > 1/3 > 0$ is a concatenation of the shortest 1-path from 1 to 3/8 with the shortest 1-path from 3/8 to 0. Note that the denominators increase until 3/8 and then decrease afterwards.

3.2.2. *Regular resolutions.* The following proposition expresses minimal regular resolutions in terms of Mac Lane valuations and shortest N-paths. We fix the following notation.

Notation 3.9. If v is a Mac Lane valuation, then $\mathcal{Y}_v$ is the v-model of $\mathbb{P}^1_K$, and $\mathcal{Y}_v^{\mathrm{reg}}$ is its minimal regular resolution. Furthermore, $\mathcal{Y}_{v,0}$ is the $\{v_0, v\}$-model of $\mathbb{P}^1_K$, and $\mathcal{Y}_{v,0}^{\mathrm{reg}}$ is its minimal regular resolution.

Proposition 3.10 ([OW18, Theorem 7.8])**.** *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$. For each i, write $\lambda_i = b_i/c_i$ in lowest terms, and let $N_i = \operatorname{lcm}_{j<i} c_j = \deg(\varphi_i)$ (Corollary 2.4). Set $\lambda_0 = \lfloor \lambda_1 \rfloor$, as well as $N_0 = N_1 = 1$ and $e(v_0/v_{-1}) = 1$. Then the minimal regular resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of $\mathcal{Y}_v$ is the normal model of $\mathbb{P}^1_K$ that includes exactly the following set of valuations:*

- *For each $1 \leq i \leq n$, the valuations*

$$v_{i,\lambda} := [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{i-1}(\varphi_{i-1}) = \lambda_{i-1},\, v_i(\varphi_i) = \lambda],$$

  as λ ranges through the shortest N_i-path from β_i to λ_i, where β_i is the least rational number greater than or equal to λ_i in $(1/N_i)\mathbb{Z} = \Gamma_{v_{i-1}}$. In other words, $\beta = \lceil N_i \lambda_i \rceil / N_i$.
- *For each $0 \leq i \leq n-1$, the valuations*

$$w_{i,\lambda} := [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_i(\varphi_i) = \lambda_i,\; v_{i+1}(\varphi_{i+1}) = \lambda],$$

  as λ ranges through the shortest N_{i+1}-path from λ_{i+1} to $e(v_i/v_{i-1})\lambda_i$, excluding the endpoints.
- *The valuation $\tilde{v}_0 := [v_0,\, v_1(\varphi_1) = \lambda_0]$ (which is just v_0 if $\lambda_1 < 1$).*

Remark 3.11. For $\lambda = e(v_i/v_{i-1})\lambda_i$, one sees that $w_{i,\lambda} = v_i$.

Remark 3.12. For v as in Proposition 3.10, consider the set S of valuations included in the minimal regular resolution $\mathcal{Y}_v^{\mathrm{reg}}$ of the v-model $\mathcal{Y}_v$ of $\mathbb{P}^1_K$. Using the partial order $\prec$ on S, one constructs a tree whose vertices are the elements of S and where there is an edge between two vertices w and w' if and only if $w \prec w'$ and there is no w'' with $w \prec w'' \prec w'$. One can show that this tree is the dual graph of $\mathcal{Y}_v^{\mathrm{reg}}$ by a repeated application of Proposition 3.2. This graph is shown in Figure 1.

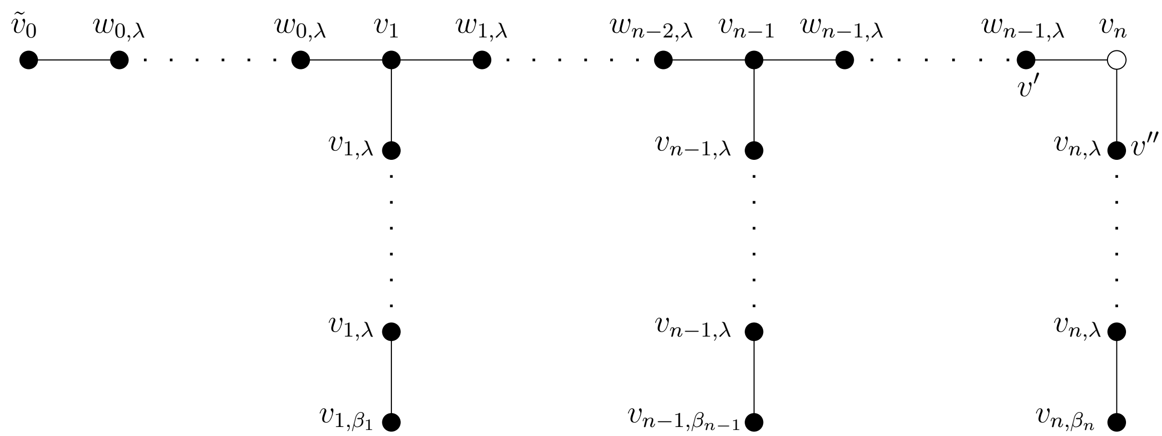

FIGURE 1. The dual graph of the minimal resolution of the $v = v_n$-model of $\mathbb{P}^1_K$. The white vertex corresponds to the strict transform of the v. The vertex labeled v' (resp. v'') corresponds to the successor (resp. precursor) valuation of v_n, see §3.3.

Corollary 3.13. *Then the valuations included in $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ are the valuations included in $\mathcal{Y}^{\mathrm{reg}}_{v}$ as well as v_0 and the valuations $[v_0,\, v_1(\varphi_1) = \lambda]$ for $\lambda \in \{1, 2, \ldots, \lambda_0 - 1\}$. Equivalently, the valuations included in $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ are exactly the valuations we would get from Proposition 3.10 if we changed our convention from $\lambda_0 = \lfloor \lambda_1 \rfloor$ to $\lambda_0 = 0$.*

Proof. If $\lambda_0 = 0$, then $\mathcal{Y}^{\mathrm{reg}}_{v}$ includes v_0, so $\mathcal{Y}^{\mathrm{reg}}_{v} = \mathcal{Y}^{\mathrm{reg}}_{v,0}$. If $\lambda_0 \geq 1$, then if $\mathcal{Z}$ is the normal model of $\mathbb{P}^1_K$ including the valuations included in $\mathcal{Y}$ as well as v_0, then there may be a singularity where the components corresponding to v_0 and $[v_0,\, v_1(\varphi_1) = \lambda_0]$ cross. Since v_0 and $[v_0,\, v_1(\varphi_1) = 0]$ are the same valuation, and since $\lambda_0 > \lambda_0 - 1 > \cdots > 1 > 0$ is the shortest 1-path from λ_0 to 0, [OW18, Corollary 7.5] shows that resolving this singularity yields exactly the description of $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ in the statement of the corollary. The equivalent description is clear, since $\lambda_1 < 1$ is equivalent to $\lambda_0 = 0$. □

Proposition 3.14. *Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$ be a Mac Lane valuation. Let $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ be the minimal regular resolution of the $\{v, v_0\}$-model of $\mathbb{P}^1_K$. If w is a valuation included in $\mathcal{Y}^{\mathrm{reg}}_{v,0}$, then $e(w/v_0) \leq e(v/v_0)$, and furthermore, if $e(w/v_0) = e(v/v_0)$, then $w \preceq v$.*

Proof. For a contradiction, choose a valuation w such that $e(w/v_0)$ is maximal among those w violating the proposition, and among these choose w such that w is maximal under $\preceq$.

First observe that, since $e(v_i/v_0) \leq e(v_n/v_0)$ for all $i \leq n$ and $v_i \preceq v_n$, we may assume

$$w \neq v_i \quad \text{for any } i. \tag{3.15}$$

Let c_w be the self-intersection number of the w-component of $\mathcal{Y}^{\mathrm{reg}}_{v,0}$. Since $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ is the minimal regular resolution of the $\mathcal{Y}_{v,v_0}$-model, and since $w \notin \{v, v_0\}$ by (3.15), we have $c_w \neq -1$, thus $c_w \leq -2$. By standard intersection theory on regular arithmetic surfaces (e.g., [OW18, (3.4)]), we have

$$-c_w e(w/v_0) = \sum_{w'} e(w'/v_0),$$

where the sum is taken over all w' such that the w'-component intersects the w-component. Since $w \neq v_i$ for any i by (3.15), Figure 1 shows that there are at most two such w'.

Since $-c_w \geq 2$ and by assumption $e(w/v_0) \geq e(w'/v_0)$ for all w' in the sum, we find that there are exactly two w' and $e(w'/v_0) = e(w/v_0)$ for each of them. By Remark 3.12, one of the w' satisfies $w \prec w'$. Since w is maximal under $\prec$, we conclude that w' does not violate the proposition. But $w \prec w'$ and $e(w/v_0) = e(w'/v_0)$ imply that w does not violate the proposition either, a contradiction. □

3.3. **Valuations related to a given Mac Lane valuation.** Let $v = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_n(\varphi_n) = \lambda_n]$. Let $N_n = \operatorname{lcm}(c_1, \ldots, c_{n-1}) = \deg(\varphi_n)$ (Corollary 2.4). We assume that $n \geq 1$ and $\lambda_n \notin \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$.

Let $\mathcal{Y}_v$ be the v-model of $\mathbb{P}^1_K$, and let $\mathcal{Y}_v^{\mathrm{reg}}$ be its minimal regular resolution. By Proposition 3.10, the following Mac Lane valuations are included in $\mathcal{Y}_v^{\mathrm{reg}}$:

- $v' := w_{n-1,\lambda'} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n'(\varphi_n) = \lambda']$,
- $v'' := v_{n,\lambda''} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n''(\varphi_n) = \lambda'']$,

where λ' is the entry directly following λ_n in the shortest N_n-path from λ_n to $e(v_{n-1}/v_{n-2})\lambda_{n-1}$, and λ'' is the entry directly preceeding λ_n in the shortest N_n-path from $\lceil N_n\lambda_n \rceil / N_n$ to λ_n. The valuation v' (resp. v'') is called the *successor* (resp. *precursor*) valuation to v.

Let us also write

- $v^* := w_{n-1,\lambda^*} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n'(\varphi_n) = \lambda^*]$,
- $v^{**} := v_{n,\lambda^{**}} = [v_0,\, v_1(\varphi_1) = \lambda_1, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n''(\varphi_n) = \lambda^{**}]$,

where v^* (resp. v^{**}) is an *arbitrary* choice among the $w_{n-1,\lambda}$ (resp. the $v_{n,\lambda}$) from Proposition 3.10.

Since $\lambda_n \notin (1/N_n)\mathbb{Z}$, we have $\lfloor N_n\lambda_n \rfloor \leq N_n\lambda' < N_n\lambda_n < N_n\lambda'' \leq \lceil N_n\lambda_n \rceil$, the first inequality coming from [OW18, Corollaries A.7, A.11]. Write $\tilde{\lambda}_n$ (resp. $\tilde{\lambda}'$, $\tilde{\lambda}''$, $\tilde{\lambda}^*$, $\tilde{\lambda}^{**}$) for $N_n\lambda_n - \lfloor N_n\lambda_n \rfloor$ (resp. $N_n\lambda' - \lfloor N_n\lambda_n \rfloor$, $N_n\lambda'' - \lfloor N_n\lambda_n \rfloor$, $N_n\lambda^* - \lfloor N_n\lambda_n \rfloor$, $N_n\lambda^{**} - \lfloor N_n\lambda_n \rfloor$). Then we obtain

$$0 \leq \tilde{\lambda}' < \tilde{\lambda}_n < \tilde{\lambda}'' \leq 1.$$

Proposition 3.16. *Let e, e', e'', e^*, and e^{**} be the denominators of $\tilde{\lambda}_n$, $\tilde{\lambda}'$, $\tilde{\lambda}''$, $\tilde{\lambda}^*$, and $\tilde{\lambda}^{**}$, respectively.*

(i) *The number $\tilde{\lambda}'$ immediately follows $\tilde{\lambda}_n$ in the shortest 1-path from $\tilde{\lambda}_n$ to 0.*
(ii) *The number $\tilde{\lambda}''$ immediately preceeds $\tilde{\lambda}_n$ in the shortest 1-path from 1 to $\tilde{\lambda}_n$.*
(iii) *The number $\tilde{\lambda}^*$ is on the shortest 1-path from $\tilde{\lambda}_n$ to 0.*
(iv) *The number $\tilde{\lambda}^{**}$ is on the shortest 1-path from 1 to $\tilde{\lambda}_n$.*

Proof. By [OW18, Lemma A.7], $N_n\lambda'$ immediately follows $N_n\lambda_n$ in the shortest 1-path from $N_n\lambda_n$ to $N_n e(v_i/v_{i-1})\lambda_{n-1}$, and thus in the shortest 1-path from $N_n\lambda_n$ to $\lfloor N_n\lambda_n \rfloor$ by [OW18, Lemma A.11]. Since translating by an integer preserves shortest 1-paths, subtracting $\lfloor N_n\lambda_n \rfloor$ from all entries of these paths yields part (i). Part (ii) follows similarly, using that $N_n\lambda''$ immediately precedes $N_n\lambda_n$ in the shortest N_n-path from $\lceil N_n\lambda_n \rceil$ to $N_n\lambda_n$. The proofs of parts (iii) and (iv) are essentially the same as the proofs of parts (i) and (ii), respectively. □

Example 3.17. If $\tilde{\lambda}_n = 3/8$, we would have $\tilde{\lambda}' = 1/3$ and $\tilde{\lambda}'' = 2/5$ (cf. Example 3.8). We could take $\tilde{\lambda}^*$ to be $1/3$ or 0, and we could take $\tilde{\lambda}^{**}$ to be $2/5$, $1/2$, or 1.

Corollary 3.18. *Let e, e', e'', e^*, and e^{**} be as in Proposition 3.16. Then*

(i) $\lambda_n - \lambda' = 1/(N_n ee')$.
(ii) $\lambda'' - \lambda_n = 1/(N_n ee'')$.
(iii) $\lambda_n - \lambda^* \geq 1/(N_n ee^*)$, *with equality if and only if* $\lambda^* = \lambda'$.
(iv) $\lambda^{**} - \lambda_n \geq 1/(N_n ee^{**})$, *with equality if and only if* $\lambda^{**} = \lambda''$.

Proof. By Proposition 3.16(i) and the definition of 1-path, $\tilde{\lambda}_n - \tilde{\lambda}' = 1/(ee')$, from which part (i) follows. Part (ii) follows similarly, using Proposition 3.16(ii). To prove part (iii), note that Proposition 3.16(iii) shows that $\tilde{\lambda}^*$ is on the shortest 1-path from $\tilde{\lambda}_n$ to 0, but that $\tilde{\lambda}^*$ does not directly follow $\tilde{\lambda}_n$ on this path unless $\lambda^* = \lambda'$. The definition of shortest 1-paths shows that $\tilde{\lambda}_n - \tilde{\lambda}^* = 1/ee^*$ if and only if $\lambda^* = \lambda'$. Since $\tilde{\lambda}_n - \tilde{\lambda}^*$ is a multiple of $1/ee^*$ by common denominators, part (iii) follows. The proof of part (iv) is exactly the same, using λ^{**}, λ'', and Proposition 3.16(iv) instead of λ^*, λ', and Proposition 3.16(iii). ☐

Lemma 3.19. *Let v, v', v'', v^*, and v^{**} be as above. If e, e', e'', e^*, and e^{**} are defined as in Proposition 3.16, then $e = e(v/v_{n-1})$, $e' = e(v'/v_{n-1})$, $e'' = e(v''/v_{n-1})$, $e^* = e(v^*/v_{n-1})$, and $e^{**} = e(v^{**}/v_{n-1})$.*

Proof. By construction, e is the denominator of $N_n\lambda_n$ (and similarly for e', e'', e^*, and e^{**}). By [OW18, Lemma 5.3(ii)], $e(v/v_0) = \operatorname{lcm}(N_n, c_n)$, where c_n is the denominator of λ_n. By [OW18, Lemma A.6], this is equal to $N_n e$. Since $N_n = e(v_{n-1}/v_0)$, we have $e = e(v/v_{n-1})$. This proves the lemma for e, and the proofs for e', e'', e^*, and e^{**} are identical. ☐

4. Some regular models of $\mathbb{P}^1$ attached to a polynomial

Let $\alpha \in \mathcal{O}_{\overline{K}}$ such that $\nu_K(\alpha) > 0$ and the minimal polynomial $f(x) \in K[x]$ of α has degree at least 2. In this section, we first define a canonical Mac Lane valuation v_f attached to f. We then define certain natural contractions of the minimal regular resolution of the v_f-model, called "Type I", "Type II", or "Type III" models. These are candidate models for the horizontal divisor D_α to be regular on. We prove some technical results about these three kinds of models. These results will then be used in the next section to show that the minimal regular model on which D_α is regular is a special kind of Type I or Type II model.

4.1. The Mac Lane valuation associated to a polynomial.

Write

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_n(\varphi_n) = \lambda_n]$$

for the unique Mac Lane valuation on $K(x)$ over which f is a proper key polynomial (Proposition 2.8(iv)). As usual, write $v_0, v_1, \ldots, v_n = v_f$ for the intermediate valuations. For $1 \leq i \leq n$, write $\lambda_i = b_i/c_i$ in lowest terms. Let $N_i = \operatorname{lcm}(c_1, \ldots, c_{i-1}) = \deg(\varphi_i)$ (Corollary 2.4). Furthermore, pick once and for all a root α of f.

Remark 4.1. If the roots of f generate a tame extension, it is easy to read off the polynomials φ_i and integers λ_i from the truncations of Newton-Puiseux expansions of the roots of f with respect to some choice of uniformizer t, as we now explain. Using Proposition 2.8(iii), we see that we can take φ_i to be the minimal polynomials of the truncations of the Newton-Puiseux expansions just before there is a jump in the lcm of the denominators of the exponents in the expansion. If α is a root of f, then Corollary 2.11 shows that $\lambda_i = \nu_K(\varphi_i(\alpha)) = \sum_{\varphi_i(\beta)=0} \nu_K(\alpha - \beta)$. If $\deg(\varphi_i) = m$, then the Galois group of the splitting field of the tame extension generated by the roots of φ_i is generated by the automorphism $t^{1/m} \mapsto \zeta_m t^{1/m}$ for a primitive m^{th} root of unity ζ_m. Since the induced $\mathbb{Z}/m\mathbb{Z}$-action on the roots of φ_i is transitive, a direct computation then shows that for each root β of φ_i, the quantity $\nu_K(\alpha - \beta)$ is equal to one of the the exponents in α where the lcm of the denominators of the exponents jumps. (This is the content of [Sri19, Lemma 8.13] using the language of characteristic/jump exponents.)

For example, let $K = \mathbb{C}((t))$ and let f be the minimal polynomial of $2t - t^{5/2} + t^{8/3} - 3t^{7/2} + t^{23/6}$. Then v_f has the form

$$v_f = [v_0,\, v_1(\varphi_1) = \lambda_1, v_2(\varphi_2) = \lambda_2],$$

and we can take $\varphi_1 = x - 2t$ and φ_2 to be the minimal polynomial of $2t - t^{5/2}$, with $\lambda_1 = 5/2$ and $\lambda_2 = 5/2 + 8/3$. This example also shows that $\deg(\varphi_i)$ and the invariants λ_i contain the same information as the characteristic exponents of the Newton-Puiseux expansion of a root of f as in [Sri19, Example 8.13] in the tame case.

For the rest of this section we will use the following notation.

Notation 4.2. Lemma 2.5 implies that we are in the situation of §3.3. Like in Section 3.3, let

- $v_f' = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n'(\varphi_n) = \lambda']$
- $v_f'' = [v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n'(\varphi_n) = \lambda'']$

be the successor and precursor valuations to v_f, respectively.

For simplicity, we write $e = e(v_f/v_{n-1})$, $e' = e(v_f'/v_{n-1})$, and $e'' = e(v_f''/v_{n-1})$. This is consistent with the notation in Lemma 3.19 and Proposition 3.16. We record for later usage that $e = \deg(f)/\deg(\varphi_n)$ by Lemma 2.3(iii).

4.2. **The model $\mathcal{Y}_{v_f}^{\text{reg}}$ and its contractions.** We now define the regular models of $\mathbb{P}^1_K$ that will be the focus of the rest of §4, and we state some specialization properties that will be useful for constructing our desired regular model in which D_α is regular.

Lemma 4.3. *On the model $\mathcal{Y}_{v_f}^{\text{reg}}$ the only component of the special fiber that D_α meets is the v_f-component.*

Proof. The multiplicity of the v_f-component of $\mathcal{Y}_{v_f}^{\text{reg}}$ in the special fiber is $e(v_n/v_{n-1})e(v_{n-1}/v_0)$. But $e(v_n/v_{n-1}) = \deg(f)/\deg(\varphi_n)$ by Lemma 2.3(iii) and $e(v_{n-1}/v_0) = \deg(\varphi_n)$ by Corollary 2.4. So the multiplicity is equal to $\deg(f)$.

By Proposition 2.11, $v_f(\varphi_n(\alpha)) = \lambda_n$. So by [OW18, Lemma 7.3(iii)] and Proposition 3.2, D_α intersects a regular point z on the v_f-model of $\mathbb{P}^1_K$, which is also a smooth point of the reduced special fiber by [OW18, Lemma 7.1]. By the previous line, we conclude that the point z is not part of the base locus of the projection $\mathcal{Y}_{v_f}^{\text{reg}} \to \mathcal{Y}_{v_f}$, and this proves the lemma. □

We now define three types of regular contractions of $\mathcal{Y}_{v_f}^{\text{reg}}$. We use the notation of Proposition 3.10 and Figure 1.

Definition 4.4. Fix f as in this section. The Type I, II, and III models below implicitly depend on f.

- A *Type I* model of $\mathbb{P}^1_K$ is any regular contraction of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ that includes at least one of the $v_{n,\lambda}$ and one of the $w_{n-1,\lambda}$, but does not include v_f.
- A *Type II* model of $\mathbb{P}^1_K$ is any regular contraction of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ that does not include v_f or any $v_{n,\lambda}$, but does include at least one of the $w_{n-1,\lambda}$.
- A (the) *Type III* model of $\mathbb{P}^1_K$ is the model where the v_f-component is contracted, as well as all the $v_{n,\lambda}$ and the $w_{n-1,\lambda}$, provided that this model does not have empty special fiber.

Remark 4.5. Since v_{n-1} is one of the $w_{n-1,\lambda}$, one see that the Type III model is the contraction of the v_{n-1}-component in $\mathcal{Y}^{\mathrm{reg}}_{v_{n-1}}$.

Definition 4.6.

- Given a Type I or Type II model $\mathcal{Y}$, define
$$v_f^* = [v_{n-1}, v_f^*(\varphi_n) = \lambda^*] = w_{n-1,\lambda^*},$$
where λ^* is maximal such that w_{n-1,λ^*} is included in $\mathcal{Y}$.
- Given a Type I model $\mathcal{Y}$, define
$$v_f^{**} = [v_{n-1}, v_f^{**}(\varphi_n) = \lambda^{**}] = v_{n,\lambda^{**}},$$
where λ^{**} is minimal such that $v_{n,\lambda^{**}}$ is included in $\mathcal{Y}$.
- If v_f^* (resp. v_f^{**}) is defined, define e^* (resp. e^{**}) to be the denominator of $N_n\lambda^*$ (resp. $N_n\lambda^{**}$). Note that this notation is consistent with that of Proposition 3.16.
- Given a Type III model $\mathcal{Y}$, define v'_{n-1} and v''_{n-1} to be the successor and precursor valuations to v_{n-1}, respectively.

Remark 4.7. Note that the components v_f^* and v_f^{**}-components of $\mathbb{P}^1_K$ intersect using Proposition 3.10 and Remark 3.12.

Remark 4.8. By Lemma 3.19, $e^* = e(v_f^*/v_{n-1})$ and $e^{**} = e(v_f^{**}/v_{n-1})$.

Lemma 4.9. *Let y be a point on the v_f-component of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$.*

(i) *Suppose $\mathcal{Y}$ is a Type I model, and $\tau\colon \mathcal{Y}^{\mathrm{reg}}_{v_f} \to \mathcal{Y}$ is the standard contraction. Then $\tau(y)$ lies on the intersection of the v_f^*- and v_f^{**}-components of $\mathcal{Y}$.*

(ii) *Suppose $\mathcal{Y}$ is a Type II model, and $\tau\colon \mathcal{Y}^{\mathrm{reg}}_{v_f} \to \mathcal{Y}$ is the standard contraction. Then $\tau(y)$ lies only on the v_f^*-component of $\mathcal{Y}$.*

(iii) *Suppose $\mathcal{Y}$ is the Type III model, and $\tau\colon \mathcal{Y}^{\mathrm{reg}}_{v_f} \to \mathcal{Y}$ is the standard contraction. Then $\tau(y)$ lies on the intersection of the v'_{n-1}- and v''_{n-1}-components of $\mathcal{Y}$.*

Proof. This follows from Remark 3.12 and Figure 1. □

Proposition 4.10. *Let α, f, v_f, v_f^*, v_f^{**}, v'_{n-1}, and v''_{n-1} be as in this section.*

(i) *If $\mathcal{Y}$ is a Type I model of $\mathbb{P}^1_K$, then the divisor D_α on $\mathcal{Y}$ meets the intersection of the two components of the special fiber corresponding to v_f^* and v_f^{**}.*

(ii) *If $\mathcal{Y}$ is a Type II model of $\mathbb{P}^1_K$, then the divisor D_α on $\mathcal{Y}$ intersects only the v_f^*-component of the special fiber.*

(iii) *If $\mathcal{Y}$ is the Type III model of $\mathbb{P}^1_K$, then the divisor D_α on $\mathcal{Y}$ meets the intersection of the two components of the special fiber corresponding to v'_{n-1} and v''_{n-1}.*

Proof. By Lemma 4.3, D_α meets the special fiber of $\mathcal{Y}_{v_f}^{\mathrm{reg}}$ only on the v_f-component. Parts (i), (ii), and (iii) of the proposition now follow from the respective parts of Lemma 4.9. ☐

Corollary 4.11. *Let $\mathcal{Y}$ be a Type I or Type II model of $\mathbb{P}^1_K$. Let α_n be a root of φ_n.*

(i) *Suppose $\beta \in \overline{K}$ has degree less than $\deg(\varphi_n)$ over K. Then D_α and D_β do not meet on the special fiber of $\mathcal{Y}$.*

(ii) *If $\mathcal{Y}$ is Type I, then D_α and D_{α_n} do not meet on the special fiber of $\mathcal{Y}$.*

(iii) *If $\mathcal{Y}$ is Type II or Type III, then D_α and D_{α_n} meet on the special fiber of $\mathcal{Y}$.*

Proof. By Proposition 4.10, D_α specializes to the v_f^*-component of the special fiber of $\mathcal{Y}$. By Corollary 2.4 and Lemma 3.19, the multiplicity of this component is $N_n e^* = \deg(\varphi_n)e^* \geq \deg(\varphi_n)$. So by [LL99, Lemma 5.1(a)], D_β does not specialize to this component. This proves part (i).

To prove part (ii), assume $\mathcal{Y}$ is Type I. Note that α_n is a root of φ_n, we have $\nu_K(\varphi_n(\alpha_n)) = \infty$, which does not lie between λ^* and λ^{**}. As a consequence, Corollary 3.4 and Proposition 4.10(i) show that D_α does not meet D_{α_n} on the special fiber of $\mathcal{Y}$.

To prove part (iii), it suffices to assume $\mathcal{Y}$ is Type II, since a Type III model is a contraction of a Type II model. Since both $\nu_K(\varphi_n(\alpha)) = \lambda_n$ and $\nu_K(\varphi_n(\alpha_n)) = \infty$ are greater than λ^*, Proposition 3.2 shows that they meet on the special fiber of the v_f^*-model of $\mathbb{P}^1_K$. This point is not a base point of the contraction $\mathcal{Y} \to \mathcal{Y}_{v_f^*}$, because that would violate Proposition 4.10(ii). Thus, D_α and D_{α_n} meet on $\mathcal{Y}$. ☐

Proposition 4.12. *On a Type I model $\mathcal{Y}$, we have $\lambda^{**} - \lambda^* = 1/(N_n e^* e^{**})$.*

Proof. Since $\mathcal{Y}$ is regular and the v_f^*- and v_f^{**}-components intersect, [OW18, Corollary 7.6] (with $\mathcal{X} = \mathcal{X}'$ there) shows that $\lambda^{**} > \lambda^*$ is the shortest N_n-path. By [OW18, Corollary A.7], $\tilde{\lambda}^{**} > \tilde{\lambda}^*$ is a shortest 1-path, where $\tilde{\lambda}^*$ and $\tilde{\lambda}^{**}$ are as in Proposition 3.16. By the definition of a 1-path, $\tilde{\lambda}^{**} - \tilde{\lambda}^* = 1/(e^* e^{**})$, so $\lambda^{**} - \lambda^* = 1/(N_n e^* e^{**})$. ☐

Proposition 4.13. *On a Type II model $\mathcal{Y}$, we have $\Gamma_{v_f^*} = \Gamma_{v_{n-1}}$.*

Proof. If $\mathcal{Y}$ is a Type II model, then it dominates $\mathcal{Y}_{v_f^*}^{\mathrm{reg}}$, and thus includes all the valuations therein. On the other hand, by the definition of a Type II model, $\mathcal{Y}$ does not include any valuation of the form $[v_0,\, v_1(\varphi_1) = \lambda_1,\, \ldots,\, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1},\, v_n(\varphi_n) = \lambda]$ with $\lambda > \lambda^*$. Applying Proposition 3.10 to v_f^*, this forces the β_n referred to in the first bullet point of Proposition 3.10 to equal λ^*. So $\lambda^* \in (1/N_n)\mathbb{Z} = \Gamma_{v_{n-1}}$. Since $\Gamma_{v_f^*} = [v_{n-1}, v_i(\varphi_i) = \lambda^*]$, it follows that λ^* together with $\Gamma_{v_{n-1}}$ generates $\Gamma_{v_f^*}$. Combining the previous two sentences, we get $\Gamma_{v_f^*} = \Gamma_{v_{n-1}}$. ☐

Lemma 4.14. *Let $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ be the φ_n-adic expansion of f. Let $\mathcal{Y}$ be a Type I or Type II model of $\mathbb{P}^1_K$, and let v_f^* and v_f^{**} be defined accordingly. Let $a_e = 1$.*

(i) *We have $v_f^*(f) = v_f^*(\varphi_n^e) = e\lambda^*$.*

(ii) *We have $v_f^*(a_i\varphi_n^i) > e\lambda^*$ for $0 \leq i \leq e-1$.*

(iii) *In the case of a Type I model, we have $v_f^{**}(f) = v_f^{**}(a_0) = e\lambda_n$.*

(iv) *In the case of a Type I model, we have $v_f^{**}(a_i\varphi_n^i) > e\lambda_n$ for $1 \leq i \leq e$.*

Proof. By Lemma 2.3(ii), φ_n^e is a term in the φ_n-adic expansion of f with minimal v_f-valuation. It is also the term whose valuation is decreased the most when v_f is replaced with

v_f^*. Thus φ_n^e is the unique term in the φ_n-adic expansion of f with minimal v_f^*-valuation. Since $v_f^*(\varphi_n) = \lambda^*$ by definition, this proves parts (i) and (ii).

Similarly, by Lemma 2.3(ii), a_0 is a term in the φ_n-adic expansion of f with minimal v_f-valuation. It is also the term whose valuation is increased the least when v_f is replaced by v_f^{**}. Thus a_0 is the unique term in the φ_n-adic expansion of f with minimal v_f^{**}-valuation. Since $v_f^{**}(a_0) = v_f(a_0) = e\lambda_n$ (Lemma 2.3(ii)), this proves parts (iii) and (iv). $\square$

Proposition 4.15. *Let $\mathcal{Y}$ be a Type I, Type II, or Type III model of $\mathbb{P}^1_K$, and let v_f^* and v_f^{**} be defined accordingly.*

- (i) *If $\mathcal{Y}$ is Type I, the quantity $b := e(\lambda_n - \lambda^*)/(\lambda^{**} - \lambda^*)$ is an integer. Furthermore, there exists a monomial s in $\varphi_1, \ldots, \varphi_{n-1}$ over K such that the divisor D_α is locally cut out by sf/φ_n^b.*
- (ii) *If $\mathcal{Y}$ is Type II, there exists a monomial t in $\varphi_1, \ldots, \varphi_{n-1}$ such the divisor D_α is locally cut out by sf, where $s = t^e$.*
- (iii) *If $\mathcal{Y}$ is Type III, then there exists $s \in K(x)$ such that the divisor D_α is locally cut out by sf, and such that the support of s is locally (near D_α) contained in the special fiber of $\mathcal{Y}$.*

Remark 4.16. Since $\varphi_1, \ldots, \varphi_{n-1}$ all have degree lower than $\deg(\varphi_n)$, Corollary 4.11(i) shows that the support of s is locally (near D_α) contained in the special fiber of $\mathcal{Y}$ in parts (i) and (ii), as well as part (iii).

Proof of Proposition 4.15. To prove the first assertion of part (i), note that $\lambda^{**} - \lambda^* = 1/(N_n e^* e^{**})$ by Proposition 4.12. So $b = N_n e e^* e^{**}(\lambda_n - \lambda^*)$. Since the denominator of λ_n divides $e(v_f/v_0) = N_n e$ and that of λ^* divides $e(v_f^*/v_0) = N_n e^*$, we have that b is an integer, and is in fact divisible by e^{**}.

Now, assume we are on a Type I model and let y be the point where D_α meets the special fiber of $\mathcal{Y}$, i.e., the specialization of $f(x) = 0$. The function f in general does not locally cut out D_α at y, because $\mathrm{div}(f)$ might also include vertical components passing through y. By Proposition 4.10(i), z is the intersection of the v_f^* and v_f^{**}-components of the special fiber. By Corollary 4.11(i), (ii), the specialization of $\varphi_i = 0$ is not y for any $1 \leq i \leq n$. So to finish the proof of part (i), it suffices to construct s as in the proposition such that $v_f^*(sf/\varphi_n^b) = v_f^{**}(sf/\varphi_n^b) = 0$.

By Lemma 4.14(i), we have $v_f^*(f/\varphi_n^b) = (e - b)\lambda^*$. Likewise, by Lemma 4.14(iii), we have $v_f^{**}(f/\varphi_n^b) = e\lambda_n - b\lambda^{**}$. Since $e^{**} \mid b$, and the denominators of λ_n and λ^{**} are $N_n e$ and $N_n e^{**}$ respectively, $e\lambda_n - b\lambda^{**} \in \Gamma_{v_{n-1}} = (1/N_n)\mathbb{Z}$. This means that there exists s as in the proposition such that $v_f^{**}(sf/\varphi_n^b) = 0$. Since $v_f^*(s) = v_f^{**}(s)$, showing that $v_f^*(sf/\varphi_n^b) = 0$ is reduced to showing that $(e - b)\lambda^* = e\lambda_n - b\lambda^{**}$. But this is immediate upon plugging in the definition of b.

Now we prove part (ii). Let y be as in part (i). By Proposition 4.10(ii), y lies on a unique component of the special fiber, namely the v_f^*-component. Furthermore, since the value group of v_f^* is $\Gamma_{v_{n-1}}$ (Proposition 4.13), we have that $v_f^*(\varphi_n) = \lambda^* \in \Gamma_{v_{n-1}}$. Thus we can find t as in the proposition such that $v_f^*(t) = -\lambda^*$. By Lemma 4.14(i), $v_f^*(t^e f) = v_f^*(sf) = 0$. By Corollary 4.11(i), the specialization of $\varphi_i = 0$ is not y for any $1 \leq i \leq n - 1$. So sf cuts out D_α, proving part (ii).

For part (iii), note that $\mathcal{Y}$ is regular, and is thus a local UFD. Since $\mathrm{div}(f)$ and D_α agree on the generic fiber in a neighborhood of D_α, there exists $s \in K(x)$ with the desired property. □

5. Minimal embedded resolution

In this section, we prove our main result, Theorem 5.17, which explicitly gives the minimal embedded resolution of $(\mathcal{X}, \mathrm{div}_0(f))$, where $\mathcal{X}$ is the v_0-model of $\mathbb{P}^1_K$ and $f \in \mathcal{O}_K[x]$ is a monic polynomial of degree at least 2. We begin in §5.1 with some general results on regularity, and then return to Mac Lane valuations and models of $\mathbb{P}^1_K$ for the proof in §5.2. The main technical lemma that makes everything work is Lemma 5.5, which depends heavily on the work in §4.

5.1. Generalities on regular models.

Lemma 5.1. *If $\mathcal{X}$ is a regular model of $\mathbb{P}^1_K$ and D is a reduced, effective, regular divisor on $\mathcal{X}$ and if $f : \mathcal{X}' \to \mathcal{X}$ is a regular modification (i.e., $\mathcal{X}'$ is regular and f is birational), then the strict transform D' of D in X' is regular.*

Proof. By [Liu02, Theorem 9.2.2], the map f is a finite sequence of blowups at reduced closed points. Since the blow up of a regular scheme at a closed point is regular ([Liu02, Lemma 8.1.19(a)]), we reduce to the case where f is a single blowup at a closed point. But then D' is a blowup of D at the same point ([TSPA, Tag 080E]), proving the lemma. □

The following proposition is well-known, but we were unable to find an exact reference. We state it only in the generality we need.

Proposition 5.2. *If $\mathcal{X}$ is a regular model of $\mathbb{P}^1_K$ and D is an integral horizontal divisor on $\mathcal{X}$, then there is a unique minimal modification $\mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is regular and the strict transform of D is regular.*

Proof. By [Liu02, Theorem 9.2.26], there exists *some* modification $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ regular under which the total transform of D has normal crossings, and in particular, the strict transform of D is thus regular. We now prove that a minimal such $\mathcal{Y}$ is unique. By [Liu02, Theorem 9.2.2], the morphism $\mathcal{Y} \to \mathcal{X}$ is a finite sequence of blowups at reduced closed points.

We now prove the proposition by induction on the minimum number n of blowups of $\mathcal{X}$ at closed points required to make the strict transform of D regular. The case $n = 0$ is trivial. If not, since blowups in centers outside $\mathrm{Supp}(D)$ do not affect D, any minimal sequence of blowups making the strict transform of D regular begins with blowing up the (unique) intersection point x of D with the special fiber of $\mathcal{X}$. Replacing $\mathcal{X}$ with its blowup at x, and noting that the strict transform of D is still integral on this blowup and then applying induction completes the proof. □

Lemma 5.3. *Let $\mathcal{Y}$ be a regular snc-model of a smooth curve Y over K, and let $y \in \mathcal{Y}$ be a closed point. Let $\mathrm{div}(f)$, $\mathrm{div}(g)$ be the divisors in* Spec $\hat{\mathcal{O}}_{\mathcal{Y},y}$ *of functions $f, g \in \hat{\mathcal{O}}_{\mathcal{Y},y}$ respectively.*

(i) *Suppose $\mathrm{div}(f)$ is of the form $\sum_{i=1}^r c_i D_i$ for some integers $c_i \geq 0$ where the D_i are Weil prime divisors. If $\sum_i c_i \geq 2$, then $f \in \mathfrak{m}^2_{\mathcal{Y},y}$.*

(ii) *Suppose* $\operatorname{div}(f) = D$ *where* D *is a Weil prime divisor corresponding to one of the irreducible components of the special fiber of* $\mathcal{Y}$ *passing through* y*. Then* $f \in \mathfrak{m}_{\mathcal{Y},y} \setminus \mathfrak{m}_{\mathcal{Y},y}^2$*.*

(iii) *If* $\operatorname{div}(f) = D$ *and* $\operatorname{div}(g) = E$*, where* D *and* E *are Weil prime divisors corresponding to two different components of the special fiber of* $\mathcal{Y}$ *passing through* Y*, then the images of* f *and* g *are linearly independent in* $\mathfrak{m}_{\mathcal{Y},y}/\mathfrak{m}_{\mathcal{Y},y}^2$*.*

Proof. First note that the regular local ring $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is a UFD and thus every height one prime ideal is principal. Thus in the situation of part (i), $f = w\prod_i f_i^{c_i}$, where w is a unit and $\operatorname{div}(f_i) = D_i$. Since the f_i lie in the maximal ideal, this proves (i).

In fact, by [CES03, Lemma 2.3.2 and its proof], we can write

$$\hat{\mathcal{O}}_{\mathcal{Y},y} \cong \mathcal{O}_K[[y_1, y_2]]/(y_1^{m_1} \cdots y_r^{m_r} - u\pi_K),$$

with $r \in \{1, 2\}$. The irreducible components of the special fiber passing through y are cut out by y_1 if $r = 1$ and, by y_1 and y_2 if $r = 2$. So in the situation of part (ii), we have $f = wy_1$ or $f = wy_2$, with w a unit. Since $\mathfrak{m}_{\mathcal{Y},y} = (y_1, y_2)$, this proves part (ii). In the situation of part (iii), we have $r = 2$, and the result follows from the fact that the images of w_1y_1 and w_2y_2 are linearly independent in $(y_1, y_2)/(y_1, y_2)^2$. □

Proposition 5.4. *Let* $\mathcal{Y}$ *be a regular model of* $\mathbb{P}^1_K$*, and let* y *be the point where* D_α *intersects the special fiber. Let* $g \in \hat{\mathcal{O}}_{\mathcal{Y},y}$ *be such that* $\operatorname{div}(g) = D_\alpha$ *on* Spec $\hat{\mathcal{O}}_{\mathcal{Y},y}$*. Then* D_α *is regular if and only if* $g \notin \mathfrak{m}_{\mathcal{Y},y}^2$*.*

Proof. This is [Liu02, Corollary 4.2.12]. □

5.2. **Non-archimedean analysis of valuations in an expansion.** Maintain our notation from §4. In particular, for the remainder of the paper, $f \in \mathcal{O}_K[x]$ is monic and irreducible of degree at least 2, α is a root of f, and on any regular model of $\mathbb{P}^1_K$, the divisor D_α is the horizontal divisor corresponding to α as in §3.1. As in §4, we use the notation v_f for the unique Mac Lane valuation over which f is a key polynomial. We also use the valuations v_f' and v_f'' from Notation 4.2, and we use the concept of Type I/II/III models associated to f from Definition 4.4, which give rise to valuations v_f^*, v_f^{**}, v_{n-1}', and v_{n-1}'' as in Definition 4.6. As in §3.1, we write $e = e(v_f/v_{n-1})$, $e' = e(v_f'/v_{n-1})$, $e'' = e(v_f''/v_{n-1})$, and, when there is a Type I/II model in play, $e^* = e(v_f^*/v_{n-1})$ and $e^{**} = e(v_f^{**}/v_{n-1})$.

We decompose the function cutting out the unique horizontal divisor of D_α using the φ_n-adic expansion of f, and analyze which of the terms in the decomposition are in $\mathfrak{m}_{\mathcal{Y},y}^2$ for Type I/II models $\mathcal{Y}$. This will be the key technical input for analyzing regularity of D_α on these models in the next section.

Lemma 5.5. *Let* $\mathcal{Y}$ *be a Type I or Type II model of* $\mathbb{P}^1_K$*, and let* $y \in \mathcal{Y}$ *be the point where* D_α *meets the special fiber of* $\mathcal{Y}$*. Let* s *be as in Proposition 4.15(i), (ii), let* b *be as in Proposition 4.15(i) if* $\mathcal{Y}$ *is Type I and let* $b = 0$ *if* $\mathcal{Y}$ *is Type II. If* $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ *is the* φ_n*-adic expansion of* f*, then we can write*

$$\frac{sf}{\varphi_n^b} = s\varphi_n^{e-b} + sa_{e-1}\varphi_n^{e-1-b} + \cdots + sa_0\varphi_n^{-b}. \tag{5.6}$$

Then,

(i) *All terms* $sa_i\varphi_n^{i-b}$ *of (5.6) for* $1 \le i \le e-1$ *are in* $\mathfrak{m}_{\mathcal{Y},y}^2$*.*

(ii) *We have* $sa_0\varphi_n^{-b} \in \mathfrak{m}_{\mathcal{Y},y}^2$ *if and only if* $v_f^* \neq v_f'$.

(iii) *We have* $s\varphi_n^{e-b} \in \mathfrak{m}_{\mathcal{Y},y}^2$ *if and only if* $\mathcal{Y}$ *is Type II or* $v_f^{**} \neq v_f''$.

(iv) *Suppose* $\mathcal{Y}$ *is Type I. If* $v_f^* = v_f'$ *and* $v_f^{**} = v_f''$, *then* $s\varphi_n^{e-b}$ *and* $sa_0\varphi_n^{-b}$ *generate linearly independent elements of* $\mathfrak{m}_{\mathcal{Y},y}/\mathfrak{m}_{\mathcal{Y},y}^2$.

Proof. Let y be the point where D_α intersects the special fiber of $\mathcal{Y}$. Recall from Proposition 4.15 that sf/φ_n^b cuts out D_α. By Remark 4.16, the horizontal part of $\mathrm{div}(s)$ does not contain y. The same is true for all of the $\mathrm{div}(a_i)$, since the a_i have degree less than φ_n by definition. Furthermore, Corollary 4.11(ii) shows that the same is true for the horizontal part of $\mathrm{div}(\varphi_n)$ if $\mathcal{Y}$ is Type I.

By Proposition 4.10, y is the intersection of the v_f^*- and v_f^{**}-components if $\mathcal{Y}$ is Type I, and y lies on only the v_f^*-component of $\mathcal{Y}$ is Type II. Write D^* and D^{**} for the prime divisors corresponding to the v_f^*- and v_f^{**}-components, respectively.

We now prove part (i). Assume $1 \leq i \leq e-1$. By Lemma 4.14(i), (ii), $v_f^*(f) < v_f^*(a_i\varphi_n^i)$, and since the divisor of sf/φ_n^b is horizontal by construction, so $0 = v_f^*(sf/\varphi_n^b) < v_f^*(sa_i\varphi_n^{i-b})$. Thus D^* lies in the support of $sa_i\varphi_n^{i-b}$. If $\mathcal{Y}$ is Type I, the same is true for D^{**} using Lemma 4.14(iii), (iv). Since no horizontal component of $\mathrm{div}(sa_i\varphi_n^{i-b})$ passes through y, we have that $sa_i\varphi_n^{i-b} \in \hat{\mathcal{O}}_{\mathcal{Y},y}$ and thus, Lemma 5.3(i) shows that $sa_i\varphi_n^{i-b} \in \mathfrak{m}_{\mathcal{Y},y}^2$. On the other hand, if $\mathcal{Y}$ is Type II, then Corollary 4.11(iii) shows that the horizontal part of $\mathrm{div}_0(\varphi_n)$ does pass through y. In this case, Lemma 5.3(ii) shows that $sa_i\varphi_n^i \in \mathfrak{m}_{\mathcal{Y},y}^2$. This concludes the proof of part (i).

For part (ii), Lemma 4.14(i), (ii) show as above that D^* is in the support of $\mathrm{div}(sa_0\varphi_n^{-b})$. If $\mathcal{Y}$ is Type I, then Lemma 4.14(iii) shows that $v_f^{**}(f) = v_f^{**}(a_0)$, so $0 = v_f^{**}(sf/\varphi_n^b) = v_f^{**}(sa_0\varphi_n^{-b})$, meaning that D^{**} is *not* in the support of $\mathrm{div}(sa_0\varphi_n^{-b})$. Observe further that the horizontal support of $\mathrm{div}(sa_0\varphi_n^{-b})$ does not pass through y, regardless of whether $\mathcal{Y}$ is Type I or Type II. This means that we have $sa_i\varphi_n^{i-b} \in \hat{\mathcal{O}}_{\mathcal{Y},y}$ and by Lemma 5.3(i), we thus have $sa_0\varphi_n^{-b} \in \mathfrak{m}_{\mathcal{Y},y}^2$ if and only if the multiplicity of D^* in $\mathrm{div}(sa_0\varphi_n^{-b})$ is at least 2.

By Corollary 2.4 and Lemma 3.19, the multiplicity of D^* in the special fiber is $N_n e^*$, so its multiplicity in $\mathrm{div}(sa_0\varphi_n^{-b})$ is $N_n e^* v_f^*(sa_0\varphi_n^{-b})$. Since $v_f^*(sf/\varphi_n^b) = 0$, $v_f^*(a_0) = v_f^{**}(a_0) = e\lambda_n$ (Lemma 4.14(iii)), and $v_f^*(f) = e\lambda^*$ (Lemma 4.14(i)), we have

$$\begin{aligned} N_n e^* v_f^*(sa_0\varphi_n^{-b}) &= N_n e^* v_f^*(a_0/f) \\ &= N_n e^*(e\lambda_n - e\lambda^*) \\ &\geq 1. \end{aligned}$$

where the inequality follows from Corollary 3.18(iii), and equality holds if and only if $v_f^* = v_f'$. So the multiplicity of D^* in $\mathrm{div}(sa_o\varphi_n^{-b})$ is at least 2 if and only if $v_f^* \neq v_f'$, finishing part (ii).

For part (iii), first suppose $\mathcal{Y}$ is Type I. Then Lemma 4.14(i), (iii), (iv) show using similar reasoning to part (ii) that D^{**} is in the support of $\mathrm{div}(s\varphi_n^{e-b})$ but D^* is not. This proves the first assertion of part (iii). Since the horizontal part of $\mathrm{div}(s\varphi_n^{e-b})$ does not pass through y, the same reasoning as in part (ii) reduces us to showing that the multiplicity of D^{**} in $\mathrm{div}(s\varphi_n^{e-b})$ is at least 2 if and only if $v_f^{**} \neq v_f''$.

By Corollary 2.4 and Lemma 3.19, the multiplicity of D^{**} in the special fiber is $N_n e^{**}$, so its multiplicity in $\mathrm{div}(s\varphi_n^{e-b})$ is $N_n e^{**} v_f^{**}(s\varphi_n^{e-b})$. Since $v_f^{**}(sf/\varphi_n^b) = 0$ and $v_f^{**}(f) = e\lambda_n$ (Lemma 4.14(i)), we have

$$\begin{aligned} N_n e^{**} v_f^{**}(s\varphi_n^{e-b}) &= N_n e^{**} v_f^{**}(\varphi_n^e/f) \\ &= N_n e^{**}(e\lambda^{**} - e\lambda_n) \\ &\geq 1. \end{aligned}$$

where the inequality follows from Corollary 3.18(iv), and equality holds if and only if $v_f^{**} = v_f''$. So the multiplicity of D^{**} in $\mathrm{div}(s\varphi_n^{e-b})$ is at least 2 if and only if $v_f^{**} \neq v_f''$, proving part (iii) in this case.

Now suppose $\mathcal{Y}$ is Type II. Then $s\varphi_n^{e-b} = s\varphi_n^e$, and by Corollary 4.11(iii), the horizontal part of $\mathrm{div}(s\varphi_n^e)$ does meet y. By Proposition 4.15, s can be taken to be an eth power in $K[x]$. Since $e \geq 2$, we have $s\varphi_n^e \in \mathfrak{m}_{\mathcal{Y},y}^e \subseteq \mathfrak{m}_{\mathcal{Y},y}^2$, finishing the proof of part (iii).

Lastly, by the proofs of parts (ii) and (iii), if $v_f^* = v_f'$ and $v_f^{**} = v_f''$, then $\mathrm{div}(s\varphi_n^{e-b}) = D^{**}$ and $\mathrm{div}(sa_0\varphi_n^{-b}) = D^*$ in Spec $\hat{\mathcal{O}}_{\mathcal{Y},y}$. Applying Lemma 5.3(iii) completes the proof of part (iv). □

Lemma 5.7. *Assume the Type III model $\mathcal{Y}$ of $\mathbb{P}_K^1$ exists. Let s be as in Proposition 4.15(iii), and write $sf = s\varphi_n^e + sa_{e-1}\varphi_n^{e-1} + \cdots + sa_0$ for the product of s with the φ_n-adic expansion of f. Then*

(i) $v_{n-1}'(s\varphi_n^e) = v_{n-1}''(s\varphi_n^e) = 0$,

(ii) $v_{n-1}'(sa_i\varphi_n^i) > 0$ *and* $v_{n-1}''(sa_i\varphi_n^i) > 0$ *for* $0 \leq i < e$.

Proof. By Proposition 4.10(iii), the divisor D_α (which is locally the same as $\mathrm{div}(sf)$) meets the intersection of the v_{n-1}'- and v_{n-1}''-components of the special fiber of $\mathcal{Y}$. Thus $v_{n-1}'(sf) = v_{n-1}''(sf) = 0$. So it suffices to show that, for $0 \leq i < e$, both $v_{n-1}'(s\varphi_n^e) < v_{n-1}'(sa_i\varphi_n^i)$ and $v_{n-1}''(s\varphi_n^e) < v_{n-1}''(sa_i\varphi_n^i)$, or equivalently, that

$$v_{n-1}'(\varphi_n^e) < v_{n-1}'(a_i\varphi_n^i) \quad \text{and} \quad v_{n-1}''(\varphi_n^e) < v_{n-1}''(a_i\varphi_n^i). \tag{5.8}$$

Fix i such that $0 \leq i < e$. We first claim that

$$v_{n-1}(\varphi_n^e) < v_{n-1}(a_i\varphi_n^i). \tag{5.9}$$

By Lemma 2.3(ii), $v_f(\varphi_n^e) \leq v_f(a_i\varphi_n^i)$. Since $\deg(a_i) < \deg(\varphi_n)$, we have $v_{n-1}(a_i) = v_f(a_i)$. On the other hand, applying Lemma 2.3(ii) to φ_n and v_{n-1} for the equality below, we have

$$v_{n-1}(\varphi_n) = e_{n-1}\lambda_{n-1} < v_f(\varphi_n),$$

where $e_{n-1} = \deg(\varphi_n)/\deg(\varphi_{n-1})$. Write $\delta = v_f(\varphi_n) - v_{n-1}(\varphi_n)$. Since $e > i$, we have

$$v_{n-1}(\varphi_n^e) = v_f(\varphi_n^e) - e\delta < v_f(\varphi_n^e) - i\delta \leq v_f(a_i\varphi_n^i) - i\delta = v_{n-1}(a_i\varphi_n^i),$$

proving (5.9).

Now, write $\varphi_n = \varphi_{n-1}^{e_{n-1}} + b_{e_{n-1}}\varphi_{n-1}^{e_{n-1}-1} + \cdots + b_0$ for the φ_{n-1}-adic expansion of φ_n, and recall from Lemma 2.3(ii) that

$$v_{n-1}(\varphi_n) = v_{n-1}(\varphi_{n-1}^{e_{n-1}}) = v_{n-1}(b_0). \tag{5.10}$$

Furthermore, the term whose valuation decreases the most upon replacing v_{n-1} with v'_{n-1} is $\varphi_{n-1}^{e_{n-1}}$, and the term whose valuation increases the least upon replacing v_{n-1} with v''_{n-1} is b_0 (since it does not increase at all). Thus,

$$v'_{n-1}(\varphi_n) = v'_{n-1}(\varphi_{n-1}^{e_{n-1}}) \text{ and } v''_{n-1}(\varphi_n) = v''_{n-1}(b_0). \tag{5.11}$$

Let c be the degree of φ_{n-1} in the φ_{n-1}-adic expansion of $a_i\varphi_n^i$, and note that

$$c < e_{n-1}e, \tag{5.12}$$

since $\deg(a_i\varphi_n^i) < \deg(\varphi_n^e)$. Then, we have

$$v'_{n-1}(\varphi_n^e) \overset{(5.11)}{=} v'_{n-1}(\varphi_{n-1}^{e_{n-1}e}) = v_{n-1}(\varphi_{n-1}^{e_{n-1}e}) - e_{n-1}e(\lambda_{n-1} - \lambda'_{n-1}) \overset{(5.10)}{=}$$
$$v_{n-1}(\varphi_n^e) - e_{n-1}e(\lambda_{n-1} - \lambda'_{n-1}) \overset{(5.9),(5.12)}{<} v_{n-1}(a_i\varphi_n^i) - c(\lambda_{n-1} - \lambda'_{n-1}) \leq v'_{n-1}(a_i\varphi_n^i)$$

and,

$$v''_{n-1}(\varphi_n^e) \overset{(5.11)}{=} v''_{n-1}(b_0^e) = v_{n-1}(b_0^e) \overset{(5.12)}{=} v_{n-1}(\varphi_n^e) < v_{n-1}(a_i\varphi_n^i) \leq v''_{n-1}(a_i\varphi_n^i).$$

This proves (5.8), and thus the lemma. □

5.3. **The minimal embedded resolution.** In this subsection, we prove the main theorem.

Proposition 5.13. *If $\mathcal{Y}$ is the Type III model of $\mathbb{P}^1_K$, then D_α is not regular on $\mathcal{Y}$.*

Proof. By Proposition 4.10(iii), D_α meets the intersection y of the v'_{n-1}- and v''_{n-1}-components of the special fiber of $\mathcal{Y}$. Let D' and D'' be the respective corresponding Weil prime divisors on $\mathcal{Y}$.

Let s be as in Lemma 5.7. Write $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \ldots + a_0$, and set $a_e = 1$. By Proposition 4.15(iii), sf cuts out D_α locally, so by Proposition 5.4, it suffices to show that $sa_i\varphi_n^i \in \mathfrak{m}^2_{\mathcal{Y},y}$ for $0 \leq i \leq e$. By Lemma 5.7, neither D' nor D'' appears with a negative coefficient in any $\operatorname{div}(sa_i\varphi_n^i)$.

Recall that in Spec $\hat{\mathcal{O}}_{\mathcal{Y},y}$, the support of s is contained in the special fiber and, by Corollary 4.11(iii), y is in the support of the horizontal part D_{α_n} of $\operatorname{div}(\varphi_n)$. Since $e \geq 2$, the divisor of $s\varphi_n^e$ is at least $eD_{\alpha_n} \geq 2D_{\alpha_n}$. By Lemma 5.3(i), $s\varphi_n^e \in \mathfrak{m}^2_{\mathcal{Y},y}$. If $0 \leq i \leq e-1$, Lemma 5.7(ii) shows that both D' and D'' lie in the support of $\operatorname{div}(sa_i\varphi_n^i)$. We again use Lemma 5.3(i) to conclude that $sa_i\varphi_n^i \in \mathfrak{m}^2_{\mathcal{Y},y}$. □

Corollary 5.14. *If $\mathcal{Y}$ is a non-trivial regular contraction of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$, then $\mathcal{Y}$ is Type I or Type II.*

Proof. Suppose $\mathcal{Y}$ is a non-trivial regular contraction of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$ that is not Type I or Type II. Then in the language of Proposition 3.10 and Corollary 3.13 applied to v_f, the model $\mathcal{Y}$ includes none of the $w_{n-1,\lambda}$ or $v_{n,\lambda}$. Thus $\mathcal{Y}$ is dominated by the unique Type III model $\mathcal{Z}$ of $\mathbb{P}^1_K$, given that $\mathcal{Z}$ includes exactly those valuations in $\mathcal{Y}^{\mathrm{reg}}_{v_n,0}$ that are not among the $w_{n-1,\lambda}$ or $v_{n,\lambda}$. By Proposition 5.13, D_α is not regular on $\mathcal{Z}$. By Lemma 5.1, D_α is therefore not regular on any regular contraction of $\mathcal{Z}$, which finishes the proof. □

Proposition 5.15. *Suppose $\mathcal{Y}$ is a nontrivial regular contraction of $\mathcal{Y}^{\mathrm{reg}}_{v_f}$. Then D_α is regular on $\mathcal{Y}$ if and only if $\mathcal{Y}$ includes v'_f or v''_f.*

Proof. By Corollary 5.14, we may assume that $\mathcal{Y}$ is either Type I or Type II. We show that if $\mathcal{Y}$ is Type I (resp. Type II), then D_α is regular on $\mathcal{Y}$ if and only if $\mathcal{Y}$ includes v'_f or v''_f (resp. v'_f). This yields the proposition.

Let $y \in \mathcal{Y}$ be the point where D_α meets the special fiber, and let s, b, and the a_i be as in Lemma 5.5. By Propositions 4.15 and 5.4, D_α being regular is equivalent to $sf/\varphi_n^b \notin \mathfrak{m}_{\mathcal{Y},y}^2$. By Lemma 5.5(i), this is equivalent to $s\varphi_n^{e-b} + sa_0\varphi_n^{-b} \notin \mathfrak{m}_{\mathcal{Y},y}^2$. By Lemma 5.5(ii), (iii), $s\varphi_n^{e-b} + sa_0\varphi_n^{-b} \notin \mathfrak{m}_{\mathcal{Y},y}^2$ implies either $v_f^* = v'_f$, or $\mathcal{Y}$ is Type I and $v_f^{**} = v''_f$. If $\mathcal{Y}$ is Type II, the reverse implication also follows from Lemma 5.5(ii), (iii), and if $\mathcal{Y}$ is Type I, the reverse implication follows from Lemma 5.5(iv). We have shown that D_α is regular if and only if $v_f^* = v'_f$ or $\mathcal{Y}$ is Type I and $v_f^{**} = v''_f$. By the definition of v_f^* and Type I/II models, $v_f^* = v'_f$ is equivalent to v'_f being included in $\mathcal{Y}$. Likewise, if $\mathcal{Y}$ is Type I, then $v_f^{**} = v''_f$ is equivalent to $\mathcal{Y}$ including v''_f. This finishes the proof $\square$

Since $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$ is a blowup of $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$ (and similarly for $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$), the following corollary is immediate.

Corollary 5.16. *The divisor D_α is regular on $\mathcal{Y}^{\mathrm{reg}}_{v'_f}$, $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$, $\mathcal{Y}^{\mathrm{reg}}_{v''_f}$, and on $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$.*

We now have the main result of the paper.

Theorem 5.17. *Let $f \in \mathcal{O}_K[x]$ be a monic irreducible polynomial of degree ≥ 2, and let $\mathcal{X}$ be the v_0-model of $\mathbb{P}^1_K$. Let v_f be the unique Mac Lane valuation over which f is a key polynomial, and let v'_f and v''_f be the valuations defined in Notation 4.2. For any Mac Lane valuation v, let $\mathcal{Y}^{\mathrm{reg}}_{v,0}$ be defined as in Notation 3.9.*

(i) *If $e(v'_f/v_0) \leq e(v''_f/v_0)$, then the minimal embedded resolution of $(\mathcal{X}, \mathrm{div}_0(f))$ is $c\colon \mathcal{Y}^{\mathrm{reg}}_{v'_f,0} \to \mathcal{X}$, where c is the canonical contraction.*

(ii) *If $e(v'_f/v_0) > e(v''_f/v_0)$, then the minimal embedded resolution of $(\mathcal{X}, \mathrm{div}_0(f))$ is $c\colon \mathcal{Y}^{\mathrm{reg}}_{v''_f,0} \to \mathcal{X}$, where c is the canonical contraction.*

Proof. By Corollary 5.16, both $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0} \to \mathcal{X}$ and $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0} \to \mathcal{X}$ are embedded resolutions of $\mathrm{div}_0(f)$. Since $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$ and $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$ are both contractions of $\mathcal{Y}^{\mathrm{reg}}_{v_f,0}$, the minimal embedded resolution is as well. By Corollary 5.14, the minimal embedded resolution includes either v'_f or v''_f. It obviously includes v_0 as well, so it is either $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$ or $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$. In particular, one of these models dominates the other, and the dominated one is the minimal embedded resolution.

Suppose $e(v'_f/v_0) \leq e(v''_f/v_0)$ as in part (i). Since $v'_f \prec v''_f$, Proposition 3.14 applied to v'_f shows that v''_f is not included in $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$, which shows that $\mathcal{Y}^{\mathrm{reg}}_{v'_f,0}$ is the dominated one, thus proving the theorem. If $e(v'_f/v_0) > e(v''_f/v_0)$ as in part (ii), then the same proposition applied to v''_f shows that v'_f is not included in $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$, showing that $\mathcal{Y}^{\mathrm{reg}}_{v''_f,0}$ is the dominated one, again proving the theorem. $\square$

Example 5.18. If f is Eisenstein, then $v_f = [v_0,\, v_1(x) = 1/\deg(f)]$. Since $1/\deg(f) > 0$ is a shortest 1-path and $[v_0,\, v_1(x) = 0] = v_0$, we have $v'_f = v_0$. So if α is a root of f, then Theorem 5.17 recovers the easy-to-verify fact that D_α is regular on the v_0-model of $\mathbb{P}^1_K$.

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