Covers of curves, Ceresa cycles, and unlikely intersections

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Outline

1. Algebraic cycles and torsion loci in moduli spaces
2. From 1-cycles on Jacobians to 0-cycles on curves using covers
3. Ceresa torsion loci and unlikely intersections

X/k: a smooth projective geometrically integral curve over k

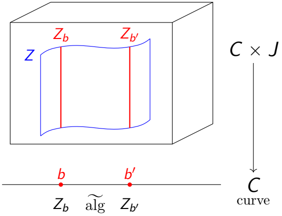
J: Jacobian of X

$\mathsf{Z}_1(J)$: Free abelian group on dimension 1 subvarieties of J

Rational/Algebraic/Homological equiv.

$\rightsquigarrow$

Corresponding filtration $(*)$ by cycles trivial with respect to $\sim$:

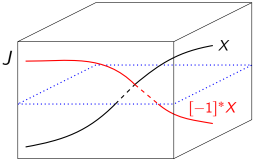


$$\mathsf{Z}_{1,\mathrm{rat}}(J) \subset \mathsf{Z}_{1,\mathrm{alg}}(J) \subset \mathsf{Z}_{1,\mathrm{hom}}(J) \subset \mathsf{Z}_1(J)$$

$C=\mathbb{P}^1$ $\quad$ $\langle Z_b - Z_{b'} \rangle$

The image of an Abel-Jacobi map

$$\begin{aligned} i_b : X &\to J \\ x &\mapsto [x-b] \end{aligned}$$



is a cycle $[X] \in Z_1(J)$.

Definition: The Ceresa cycle in $Z_1(J)$ is the 1 cycle

$$\mathrm{Cer}(X,b) := [X] - [-1]^*[X] = X - X^-.$$

$$Z_{1\,\mathrm{rat}}(J) \subset Z_{1\,\mathrm{alg}}(J) \subset Z_{1\,\mathrm{hom}}(J) \subset Z_1(J). \quad (*)$$

Question: How deep in $(*)$ does $\mathrm{Cer}(X,b)$ lie?

The Ceresa cycle is homologically trivial

Definition: The Ceresa cycle in $Z_1(J)$ is the 1 cycle

$$\mathrm{Cer}(X, b) := [X] - [-1]^*[X] = X - X^-.$$

$$Z_{1\,\mathrm{hom}}(J) := \ker\left(Z_1(J) \xrightarrow{\mathrm{cyc}_\ell} H^{2g-2}_{\text{ét}}(J_{\overline{k}}, \mathbb{Q}_\ell(g-1)) \right).$$

Fact: $[-1]^*$ acts trivially on $H^{2g-2}_{\text{ét}}(J_{\overline{k}}, \mathbb{Q}_\ell(g-1))$. $\Rightarrow$

$$\mathrm{Cer}(X, b) \in Z_{1\,\mathrm{hom}}(J).$$

The Ceresa cycle of the generic curve of $g \geqslant 3$ is algebraically nontrivial

$$\begin{aligned}\mathsf{CH}_{1\,\mathrm{hom}}(J) &:= \mathsf{Z}_{1\,\mathrm{hom}}(J)/\,\mathsf{Z}_{1\,\mathrm{rat}}(J)\\ \mathsf{Gr}_1(J) &:= \mathsf{Z}_{1\,\mathrm{hom}}(J)/\,\mathsf{Z}_{1\,\mathrm{alg}}(J).\end{aligned}$$

Theorem (Ceresa, 1983)

Let X be the generic curve of genus $g \geqslant 3$.
Then $\mathsf{Cer}(X, b)$ has infinite order in $\mathsf{Gr}_1(J)$.

Theorem (Bloch, 1984)

If X is the Fermat quartic, then $\mathsf{Cer}(X, b)$ has infinite order in $\mathsf{Gr}_1(J)$.

Conjecture (Beilinson-Bloch, 1980s)

If X is a nice genus g curve defined over a number field K, then

$$\mathrm{rank}(\mathrm{CH}_{1\,\mathrm{hom}}(J)) = \mathrm{ord}_{s=g-1}\, L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s).$$

Remark: Neither finite generation of $\mathrm{CH}_{1\,\mathrm{hom}}(J)$ nor analytic continuation of $L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s)$ to $s = g - 1$ are known!

BSD	$E(\mathbb{Q})$	$L(H^1(E_{\overline{\mathbb{Q}}}), s)$	Heegner point
Beilinson-Bloch	$\mathrm{CH}_{1\,\mathrm{hom}}(J)$	$L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s)$	Ceresa cycle?

Let b satisfy $(2g-2)b = K_X$, and let $\mathrm{Cer}(X) := \mathrm{Cer}(X, b)$.

Definition: The Ceresa torsion locus is

$$\mathcal{M}_g^{\text{tors}} := \{[X] \in \mathcal{M}_g(\mathbb{C}) \mid \mathrm{Cer}(X) \text{ is torsion in } \mathrm{CH}_{1\,\mathrm{hom}}(J)\}.$$

$\mathcal{M}_g^{\text{tors}}$ is a countable union of closed subvarieties of $\mathcal{M}_g$ for $g \geqslant 3$.

How do subvarieties $W \subset \mathcal{M}_g$ intersect $\mathcal{M}_g^{\text{tors}}$?

- Which subvarieties $W \subset \mathcal{M}_g$? $\dim W$?
- Are there explicit subvarieties W with $W \cap M_g^{\text{tors}} = \varnothing$?
  Today!

What's known about Ceresa torsion loci?

- Hyperelliptic locus $\mathcal{M}_g^{\text{hyp}} \subset \mathcal{M}_g^{\text{tors}}$.
- (1995) Collino–Pirola: If $W \subset \mathcal{M}_g$ for $g > 3$ is nonhyperelliptic and codim $W < (g+2)/3$, then $\text{Cer}(C)$ is nontorsion in $\text{Gr}_1(J)$ got a very general point $[C]$ of W.
- $\mathcal{M}_g^{\text{hyp}} \subsetneq \mathcal{M}_g^{\text{tors}}$ for small g. Key: Automorphisms.

  (Bisogno–Li–Litt–S., Qiu–Zhang, Beauville–Schoen, Laterveer, Laga–Shnidman).

  Don't know if $\mathcal{M}_g^{\text{hyp}} \subsetneq \mathcal{M}_g^{\text{tors}}$ for large g!

What's known about Ceresa torsion loci?

- (2024) Gao & Zhang, Hain, Kerr & Tayou:

  There exists an open dense "ample locus" $U_g \subset \mathcal{M}_g$ such that $U_g \cap \mathcal{M}_g^{\text{tors}}$ is an atmost countable subset of $U_g(\overline{\mathbb{Q}})$.

  Gao and Zhang proved a lower bound for the Beilinson–Bloch height of $\text{Cer}(C)$ on $U_g(\overline{\mathbb{Q}}) \implies$ Northcott holds in $U_g(\overline{\mathbb{Q}})$.

- (2023) Laga & Shnidman: Let $X_t : y^3 = x^4 + 2tx^2 + 1$.
  Then $\text{Cer}(X_t)$ is torsion if and only if the point $(\sqrt[3]{t^2-1}, t)$ on the elliptic curve $y^2 = x^3 + 1$ is torsion.

Today's theorems: Ceresa cycles in Hurwitz spaces

Theorem (Bhatnagar, Devadas, D'Nelly-Warady, S.)

Let G be a finite group. Let $\mathcal{M}_G$ denote the space of genus g curves Y admitting automorphisms by G.
If $g(Y/G) \geqslant 2$ and $Y \to Y/G$ is ramified at at least one point, then the very general curve in $\mathcal{M}_G$ has infinite order Ceresa cycle.

Theorem (Bhatnagar, Devadas, D'Nelly-Warady, S.)

Consider the 1-parameter family of genus 6 curves C_t defined by

$$y^{12} = \left(\frac{x+1}{x-1}\right)^3 \left(\frac{x+t}{x-t}\right)^4 .$$

Then $\mathrm{Cer}(C_t)$ is nontorsion for every $t \in \mathbb{C}\backslash\{0, \pm 1\}$.

Laga–Shnidman: $\mathrm{Cer}(C_t)$ torsion in $\mathrm{Gr}_1(J_t)$ for every $t \in \mathbb{C}\backslash\{0, \pm 1\}$.

Today's theorems: Ceresa cycles in Hurwitz spaces

- We also show $\exists$ explicit 2–parameter family $\mathcal{H}$ of genus 6 curves covering genus 3 curves with an S_3 action such that $\mathcal{H} \cap \mathcal{M}_g^{\mathrm{tors}}$ is Zariski closed of positive codimension in $\mathcal{H}$.
- Using the recently proven
  - relative Manin–Mumford conjecture (Gao–Habegger, 2023), &,
  - Habegger's bounded height theorem (2009),

  we show there are infinitely many such families $\mathcal{H}$ where the torsion locus is Zariski closed of positive codimension in $\mathcal{H}$.

Key dimension reduction technique: Intersection theory of algebraic cycles using correspondences coming from covers.

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Dimension reduction: Shadows of the Ceresa cycle

Definition (Ellenberg–Logan–S. '24)

Let $\phi \colon C \to C'$ be a separable degree d cover of curves with ramification divisor R_ϕ. The *relative canonical shadow* *in* $\mathrm{Pic}^0(C)$ *is*

$$D_\phi := d(2g_{C'} - 2)R_\phi - \deg(R_\phi)\phi^*(K_{C'}) + 2(dR_\phi - \phi^*\phi_* R_\phi).$$

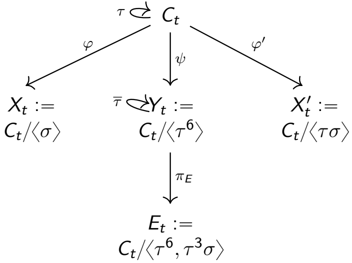
Theorem (E–L–S. '24)

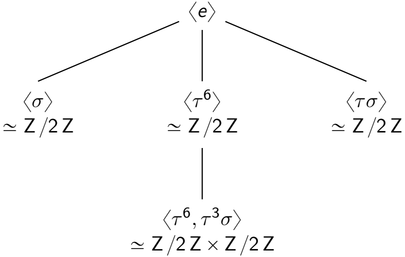
If D_ϕ is infinite order, then $\mathrm{Cer}(C)$ *has infinite order.*

- When ϕ is Galois, $2(dR_\phi - \phi^*\phi_* R_\phi) = 0$ (since $\phi^*\phi_* = [d]$).
- Observe $\phi_*(D_\phi) = d(2g_{C'} - 2)(\phi_*(R_\phi)) - d\deg(R_\phi)K_{C'}$

Proof sketch: $\mathrm{Cer}(C_t)$ is nontorsion for every $t \in \mathbb{C} \backslash \{0, \pm 1\}$

$$y^{12} = \left(\frac{x+1}{x-1}\right)^3 \left(\frac{x+t}{x-t}\right)^4, \sigma(x,y) := (-x, y^{-1}), \tau(x,y) := (x, \zeta_{12} y).$$





$$D_1(t) := (\pi_E)_* (\overline{\tau}^2)_* \psi_* (D_\varphi), \quad D_2(t) := (\pi_E)_* \psi_* (D_{\varphi'}).$$

Idea: Ceresa torsion locus is contained in the simultaneous torsion locus of $D_1(t), D_2(t)$ of E_t. Use Masser–Zannier theorem+Stoll.

Ceresa cycles of ramified covers of curves of genus $\geqslant 2$

Theorem (Bhatnagar, Devadas, D'Nelly-Warady, S.)

Let G be a finite group. Let $\mathcal{M}_G$ denote the space of genus g curves Y admitting automorphisms by G.
If $g(Y/G) \geqslant 2$ and $Y \to Y/G$ is ramified at at least one point, then the very general curve in $\mathcal{M}_G$ has infinite order Ceresa cycle.

Sketch: $\pi_*(D_\pi) = (\#G)\left((2g_{Y/G} - 2)\pi_*(R_\pi) - (\deg R_\pi)K_{Y/G}\right)$.

The divisor $\pi_*(D_\pi)$ is an Abel–Jacobi map applied to branch points.

Manin–Mumford conjecture a.k.a. Raynaud's theorem: For very general branch points, $\pi_*(D_\pi)$ (hence $\mathrm{Cer}(Y)$) has infinite order.