Certifying nontriviality of Ceresa classes of curves

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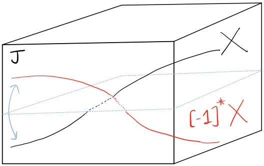
X/k:	smooth projective geometrically integral curve over a field k
b:	k-rational point of X
J:	Jacobian of X
$\mathrm{Z}^{g-1}(J)$:	Group of codimension $g-1$ cycles on J defined over k

$$\begin{aligned} i\colon X &\to J \\ x &\mapsto [x-b] \end{aligned}$$

The Ceresa cycle $\mathrm{C}(X, b)$ in $\mathrm{Z}^{g-1}(J)$ is the canonical codimension $g-1$ cycle:

$$\mathrm{C}(X, b) := [X] - [-1]^*[X].$$



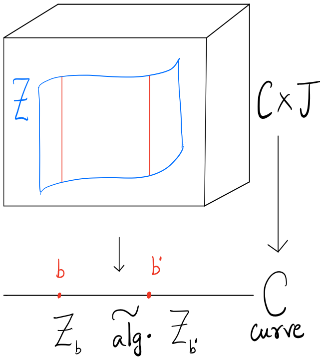
Equivalence relations on algebraic cycles

Rational/Algebraic/Homological equiv.

$\rightsquigarrow$

Corresponding filtration $(*)$ by cycles trivial with respect to $\sim$:

$$\mathsf{Z}^{g-1}_{\mathrm{rat}}(J) \subset \mathsf{Z}^{g-1}_{\mathrm{alg}}(J) \subset \mathsf{Z}^{g-1}_{\mathrm{hom}}(J) \subset \mathsf{Z}^{g-1}(J).$$



Question: How deep in $(*)$ does the Ceresa cycle lie?

The Ceresa cycle is homologically trivial

Definition: The Ceresa cycle in $Z^{g-1}(J)$ is the codim. $g-1$ cycle

$$C(X,b) := [X] - [-1]^*[X] = X - X^-.$$

$$Z^{g-1}_{\mathrm{hom}}(J) := \ker\left(Z^{g-1}(J) \xrightarrow{\mathrm{cyc}_\ell} H^{2g-2}_{\text{ét}}(J_{\overline{k}}, \mathbb{Q}_\ell(g-1))\right).$$

Fact: $[-1]^*$ acts trivially on $H^{2g-2}_{\text{ét}}(J_{\overline{k}}, \mathbb{Q}_\ell(g-1))$. $\Rightarrow$

$$C(X,b) \in Z^{g-1}_{\mathrm{hom}}(J).$$

The Ceresa cycle of the generic curve of $g \geqslant 3$ is algebraically nontrivial

Fact:
X hyperelliptic, b Weierstrass point $\Rightarrow C(X,b) = 0 \in \mathsf{Z}^{g-1}(J)$.

Theorem (Ceresa, 1983)

Let X be the generic curve of genus at least 3.
Then every multiple of $\mathsf{C}(X,b)$ is not in $\mathsf{Z}^{g-1}_{\mathrm{alg}}(J)$.

Theorem (B.Harris, 1983)

Let X be the Fermat quartic. Then $\mathsf{C}(X,b)$ is not in $\mathsf{Z}^{g-1}_{\mathrm{alg}}(J)$.

Examples where the Ceresa cycle has infinite order mod. $\sim$

$$Z^{g-1}_{\mathrm{rat}}(J) \subset Z^{g-1}_{\mathrm{alg}}(J) \subset Z^{g-1}_{\mathrm{hom}}(J) \subset Z^{g-1}(J).$$

Author(s)	Year	Curve(s)	Equiv. $\sim$
Bloch	1984	Fermat quartic	Alg.
Kimura	2000	Certain Fermat quotients	Alg.
Tadokoro	2009	Klein quartic, Fermat curve C_6	Alg.
Otsubo	2012	Fermat curve C_n, for $n \leqslant 1000$	Alg.
Eskandari-Murty	2021	Fermat curve C_n, $\exists$ prime $p \geqslant 7$, $p \mid n$	Rat.
Ellenberg-Logan-S.-Venkatesh	2023+	$254,602$ smooth plane quartics	Rat.

Define $\mathsf{CH}^{g-1}_{\mathrm{hom}}(J) := \mathsf{Z}^{g-1}_{\mathrm{hom}}(J)/\,\mathsf{Z}^{g-1}_{\mathrm{rat}}(J)$.

Conjecture (Beilinson-Bloch, 1980s)

If X is a nice genus g curve defined over a number field K, then

$$\mathrm{rank}(\mathsf{CH}^{g-1}_{\mathrm{hom}}(J)) = \mathrm{ord}_{s=g-1} L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s).$$

Remark: Neither the finite generation of $\mathsf{CH}^{g-1}_{\mathrm{hom}}(J)$ nor the analytic continuation of $L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s)$ to $s = g-1$ are known!

BSD	$E(\mathbb{Q})$	$L(H^1(E_{\overline{\mathbb{Q}}}), s)$	Heegner point
Beilinson-Bloch	$\mathsf{CH}^{g-1}_{\mathrm{hom}}(J)$	$L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s)$	Ceresa cycle?

Question: When $\mathrm{ord}_{s=g-1} L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s) \geqslant 1$, can we produce an element of $\mathrm{Z}^{g-1}_{\mathrm{hom}}(J)$ with infinite order in $\mathrm{CH}^{g-1}_{\mathrm{hom}}(J)$?

Question: When $\mathrm{ord}_{s=g-1} L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s) \geqslant 1$, can we produce an element of $\mathrm{Z}^{g-1}_{\mathrm{hom}}(J)$ with infinite order in $\mathrm{CH}^{g-1}_{\mathrm{hom}}(J)$?

Natural candidate algebraic cycle: The Ceresa cycle $\mathrm{C}(X, b)$.

Question: When $\mathrm{ord}_{s=g-1} L(H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell), s) \geqslant 1$, can we produce an element of $\mathrm{Z}^{g-1}_{\mathrm{hom}}(J)$ with infinite order in $\mathrm{CH}^{g-1}_{\mathrm{hom}}(J)$?

Natural candidate algebraic cycle: The Ceresa cycle $\mathrm{C}(X, b)$.

Today:
An algorithm that certifies $\mathrm{C}(X, b)$ has infinite order, or does not terminate.

For e.g.,
when run on $254,704$ low height smooth plane quartics $/\mathbb{Q}$, algo. failed to terminate (i.e. certify having infinite order) in 102 cases.

Studying the Ceresa cycle using ℓ-adic cycle class maps

Want: An algorithm for certifying that $\mathrm{C}(X, b)$ has infinite order.

Tool: The (refined) ℓ-adic cycle class map $\widetilde{\mathrm{cyc}}_\ell$:

$$Z^{g-1}_{\mathrm{hom}}(J) \xrightarrow{\widetilde{\mathrm{cyc}}_\ell} H^1(G_K, H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell))$$

The Ceresa class $\nu(X) := (\widetilde{\mathrm{cyc}}_\ell(\mathrm{C}(X, b)))_\ell$.[1]

Easier goal:
An algorithm for certifying $\nu(X)$ has infinite order.

[1] We replace the G_K-module $H^{2g-3}_{\text{ét}}(J_{\overline{K}}, \mathbb{Q}_\ell)$ by a canonical quotient M to eliminate dependence on the base point b.

Recall that
$\nu_\ell(X) := \widetilde{\mathrm{cyc}}_\ell(\mathrm{C}(X,b)) \in H^1(G_K, M),\ \nu(X) = (\nu_\ell(X))_\ell$.

Theorem (Ellenberg-Logan-S.-Venkatesh)

Let X be a nice curve defined over a number field K. Let $\mathfrak{p}$ be a good prime for X. Define

$$D := \left[\#X(\mathbb{F}_\mathfrak{p}) \sum_{P \in X(\mathbb{F}_{\mathfrak{p}^2})} P - \#X(\mathbb{F}_{\mathfrak{p}^2}) \sum_{P \in X(\mathbb{F}_\mathfrak{p})} P\right] \in J(\mathbb{F}_\mathfrak{p}).$$

Define nonzero integers $N, N_\mathfrak{p}$ by

$$N := \det(\mathrm{Frob}_\mathfrak{p} - I)|_M, \quad \text{and}, \quad N_\mathfrak{p} := \mathrm{ord}_{J(\mathbb{F}_\mathfrak{p})} D.$$

If the Ceresa class $\nu(X)$ is torsion, then $N_\mathfrak{p} \mid N$.

An algorithm for certifying the Ceresa class has infinite order

$$N := \det(\mathrm{Frob}_{\mathfrak{p}} - I)|_M, \quad \text{and}, \quad N_{\mathfrak{p}} := \mathrm{ord}_{J(\mathbb{F}_{\mathfrak{p}})} D.$$

Theorem (Ellenberg-Logan-S.-Venkatesh)

If the Ceresa class $\nu(X)$ is torsion, then $N_{\mathfrak{p}} \mid N$.

Algorithm:

Compute N and $N_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ up to a chosen bound.

If $\exists \mathfrak{p}$ such that $N_{\mathfrak{p}} \nmid N$, then output that $\nu(X)$ has infinite order.

Theorem proof ingredients:

N: $(M/\ell M)^{G_K} \twoheadrightarrow H^1(G_K, M)[\ell]$, Weights $G_K \circlearrowright H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell(1))$.

$N_{\mathfrak{p}}$: Reduction modulo $\mathfrak{p}$, Frobenius correspondences.

Outline

Experiments with low height quartic curves

Preliminary code/data available at
https://github.com/padmask/CeresaCertificates

Dataset:
$255,654$ smooth plane quartics with coefficients in $\{-1, 0, 1\}$ –
Full set of reps. for $S_3 \circlearrowright$ variables, scaling each variable by ± 1.
$254,704$ $\overline{\mathbb{Q}}$-isom. classes by Diximier-Ohno invariants.

Fails to certify $\nu(X)$ having infinite order: by computing N_p for $p < 50$ in 164 examples, falling into 102 $\overline{\mathbb{Q}}$-isom. classes.
These include some of the known torsion cycle examples.

Curious fact:
All 164 examples have nontrivial geometric automorphisms!
Potentially new example of curve with torsion Ceresa cycle:
$x^4 + xz^3 + y^2z^2 + yz^3 = 0$.

Summary & Further questions

Summary: A new ℓ-adic algorithm:

INPUT: A curve X over a number field.

OUPUT: A certificate that the Ceresa class $\nu(X)$ has infinite order, or does not terminate.

Further questions:

- Do Frobenius correspondences detect the order of $\nu(X)$?

  i.e. $\exists$? curve $X/\mathbb{F}_q$ with $\operatorname{ord}\nu(X) > 1$ but $N_{\mathfrak{p}} = 1$?

- Explicit p-adic algorithm (using Coleman integration) for certifying nontriviality of $C(X)$ in the Griffiths group

  i.e. is $C(X, b)$ nontrivial modulo *algebraic* equivalence?

Examples where the Ceresa cycle is torsion modulo $\sim$

Question: Are there non-hyperelliptic curves with torsion Ceresa cycle?

Author(s)	Year	Curve(s)	Equiv. $\sim$
Bisogno-Li-Litt-S.	2020	Fricke-Macbeath curve, $g = 7$ Hurwitz curve	Coh.*
Gross	2021	Fricke-Macbeath	Rat.†
Beauville-Schoen	2021	$x^4 + xz^3 + y^3 z = 0$	Alg.
Lilienfeldt-Shnidman	2021	$y^3 = x^4 + 1$	Alg.
Qiu-Zhang	2022	Curves X with $\mathrm{Aut}(X) = G$, and $(H^1(X)^{\otimes 3})^G = \varnothing$	Rat.

* Result weaker than rational equivalence.

† Conditional on Beilinson-Bloch conjectures.