CONDUCTOR-DISCRIMINANT INEQUALITY FOR TAMELY RAMIFIED CYCLIC COVERS I

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Abstract. We prove conductor-discriminant inequalities for all $\mathbb{Z}/n$-covers of $\mathbb{P}^1$ defined over discretely valued fields K with excellent valuation ring $\mathcal{O}_K$ and perfect residue field of characteristic not dividing n, modulo some calculations appearing in work [Ste26] of the third author. Specifically, when such a curve X is given by $y^n = f(x)$ with $f(x) \in \mathcal{O}_K[x]$ and $n \mid \deg(f)$, and if $\mathcal{X}$ is its minimal regular model over $\mathcal{O}_K$, then the negative of the Artin conductor of $\mathcal{X}$ is bounded above by $(n-1)v_K(\operatorname{disc}(\operatorname{rad} f))$. This is a direct generalization of previous work of the first two authors on hyperelliptic curves, which in turn generalized work of Ogg, Saito, Liu, and the second author. When f is monic, this strengthens a result of Kohls stating that the conductor exponent of the Jacobian of such a curve is bounded above by $(n-1)v_K(\operatorname{disc}(\operatorname{rad} f))$.

1. Introduction

We prove conductor-discriminant inequalities for all $\mathbb{Z}/n$-covers of $\mathbb{P}^1$ defined over Henselian discretely valued fields K with excellent valuation ring[1] and perfect residue field of characteristic not dividing n. This generalizes a previous result of the first two authors ([OS24]) covering the case $n = 2$, which in turn generalized earlier work of Ogg, Saito, and Liu on genus 1 and 2 curves ([Sai88], [Liu94]).

1.1. **Main theorem.** Let K be a Henselian discretely valued field with perfect residue field k. Let $\mathcal{O}_K$ be the ring of integers of K. Assume that $\mathcal{O}_K$ is excellent. Let $\nu_K \colon K \to \mathbb{Z} \cup \{\infty\}$ be the corresponding discrete valuation. Fix a uniformizer π_K of K. Let X be a nice (i.e. smooth, projective, geometrically integral) curve of genus $g \geq 1$ defined over K. Let $\mathcal{X}$ be a proper, flat, regular $\mathcal{O}_K$-scheme with generic fiber X. The *Artin conductor* associated to the model $\mathcal{X}$ is defined by

$$\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = \chi(\mathcal{X}_{\overline{K}}) - \chi(\mathcal{X}_{\bar{k}}) - \delta(X/K),$$

where χ is the Euler characteristic for the ℓ-adic cohomology and $\delta(X/K)$ is the Swan conductor associated to the ℓ-adic representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$). The Artin conductor is a measure of degeneracy of the model $\mathcal{X}$; it is a non-positive integer that is zero precisely when $\mathcal{X}/\mathcal{O}_K$ is smooth or when $g = 1$ and $(\mathcal{X}_k)_{\text{red}}$ is smooth.

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[1] The assumption that K has excellent valuation ring is mild and holds in many common settings, e.g., whenever K is complete. We use this assumption only for Lemmas 10.3 and 10.7.

For instance, if $\mathcal{X}/\mathcal{O}_K$ is a regular, semistable model, then $-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K)$ equals the number of singular points of the special fiber $\mathcal{X}_k$.

In [Blo87, Proposition 1.1], Bloch showed that the Artin conductor appears in the functional equation for the ζ-function attached to a global proper flat regular $\mathcal{O}_K$-model $\mathcal{X}$ for a curve X defined over a number field K, analogous to how the conductor exponent of the Jacobian of X appears in the usual ζ-function attached to X. Databases of curves such as the LMFDB [LMFDB] order curves by the conductor exponent of their Jacobian, and this requires algorithms to compute or at least bound this invariant, which is a difficult problem in general. In this paper, we describe an approach involving explicit regular models for $\mathbb{Z}/n$-covers of $\mathbb{P}^1_K$, generalizing previous work of the first two authors when $n = 2$. In some cases, this bound improves on known upper bounds for the conductor exponent computed using explicit semistable models (see [DDMM23] and [BW17] for explicit constructions of semistable models for hyperelliptic/superelliptic curves, and computation of related invariants, and [Koh19] for bounds on the conductor exponent for superelliptic curves).

The upper bound is in terms of a more easily computable measure of degeneracy $\Delta_{X/K}$ available to a $\mathbb{Z}/n$-cover X of $\mathbb{P}^1$ (see Definition 1.3 below). The invariant $\Delta_{X/K}$ detects degeneracy of the branch locus on reduction from the defining equation of the cover. To define $\Delta_{X/K}$, we begin by fixing an integral Weierstrass equation for X, namely an affine equation for X of the form $y^n = f(x)$ with $f(x) \in \mathcal{O}_K[x]$ that is nth power-free.

Definition 1.1. Assume $f(x) \in \mathcal{O}_K[x]$ is nth power-free and non-constant. Let $\widetilde{f}_n(x_0, x_1) := x_0^d f(x_1/x_0)$, where $d = n\lceil \deg(f)/n \rceil$; this is the homogenization of f, with extra x_0 factors if $n \nmid \deg(f)$. Let $g := \operatorname{rad}(\widetilde{f}_n) \in \mathcal{O}_K[x]/\mathcal{O}_K^\times$, where the radical of a polynomial is the product of its distinct irreducible factors. The *n-adapted discriminant* $\operatorname{disc}_n(f)$ of f is defined to be $\operatorname{disc}(g) \in \mathcal{O}_K/\mathcal{O}_K^\times$, where $\operatorname{disc}(g)$ is the homogeneous discriminant defined in [GKZ08, Ch. 12, (1.22),(1.29)].

Observe that x_0 divides g as above precisely when $x = \infty$ is a ramification point of the cover $X \to \mathbb{P}^1_K$. See Lemma 2.1 for why $\operatorname{disc}_n(f)$ detects collisions between the components of the branch divisor of the $\mathbb{Z}/n$-cover upon reduction to k.

Remark 1.2. If $n = 2$, then $\widetilde{f}_2 = \operatorname{rad}(\widetilde{f}_2) \mod \mathcal{O}_K^\times$ for f as above, and Definition 1.3 of $\operatorname{disc}_2(f)$ reduces to the definition $\operatorname{disc}'(f)$ given in [OS24].

Definition 1.3. A *minimal Weierstrass equation* is an equation for which the integer $\nu_K(\operatorname{disc}_n(f))$ is as small as possible amongst all integral Weierstrass equations for the curve X, and the *minimal discriminant* $\Delta_{X/K}$ of X is $\nu_K(\operatorname{disc}_n(f))$ for such an equation.

For a curve X as above, let $\mathcal{X}$ be the normalization of the standard model $\mathbb{P}^1_{\mathcal{O}_K} := \operatorname{Proj} \mathcal{O}_K[x_0, x_1]$ of $\mathbb{P}^1_K$ corresponding to the coordinate $x := x_1/x_0$ in the function field $K(X)$. Then $\mathcal{X}$ is a smooth model for X if and only if $\Delta_{X/K} = 0$ (see Lemma 2.1 (ii)). Conversely, if $\mathcal{X}$ is a smooth proper model for a $\mathbb{Z}/n$-cover of $\mathbb{P}^1$, then $\mathcal{X}$ arises as the normalization of $\mathbb{P}^1_{\mathcal{O}_K}$ in $K(X)$, and $\nu_K(\operatorname{disc}_n(f)) = 0$ for a polynomial f giving rise to an equation for the cover as $y^n = f(x)$ (see Proposition 2.4).

The main theorem of the paper is the following:

Theorem 1.4 (Conductor-discriminant inequality)**.** *Let K be the fraction field of an excellent, Henselian discrete valuation ring $\mathcal{O}_K$ with perfect residue field of characteristic p ($p = 0$*

is allowed), and let $f \in \mathcal{O}_K[x]$ be a nth power-free polynomial. Let X be the cyclic cover of $\mathbb{P}^1_K$ with affine equation $y^n = f(x)$. Assume $p \nmid n$ and that the genus of X is ≥ 1. Then there exists a proper flat regular $\mathcal{O}_K$-model $\mathcal{X}_f$ of X such that

$$-\operatorname{Art}(\mathcal{X}_f/\mathcal{O}_K) \leq (n-1)\nu_K(\operatorname{disc}_n(f)). \tag{1.5}$$

We call (1.5) the *conductor-discriminant inequality for f*.

If $g(X) \geq 1$, let $\operatorname{Art}(X/K)$ denote the Artin conductor associated to the *minimal* proper regular model of X over $\mathcal{O}_K$.

Corollary 1.6. *Let $X \to \mathbb{P}^1_K$ be a $\mathbb{Z}/n$-cover with $g(X) \geq 1$. Let $\Delta_{X/K}$ be the minimal discriminant of X and let $\operatorname{Art}(X/K)$ denote the Artin conductor of the minimal regular model of X. Then $-\operatorname{Art}(X/K) \leq (n-1)\Delta_{X/K}$.*

Remark 1.7. Let X be the curve with affine equation $y^n = x^n - \pi_K$. Then the normalization $\mathcal{X}$ of $\mathbb{P}^1_{\mathcal{O}_K}$ in $K(X)$ is the minimal regular model of X, and $-\operatorname{Art}(X/K) = (n-1)^2$ (see Example 8.3 for a proof). A direct computation shows that $\Delta_{X/K} = n - 1$. Hence $-\operatorname{Art}(X/K) = (n-1)\Delta_{X/K}$, showing that the factor $n-1$ multiplying $\Delta_{X/K}$ in Corollary 1.6 is optimal for any n prime to p.

In fact, a similar inequality to 1.6 was obtained by Kohls in his thesis ([Koh19, Theorem 5.1.11]) with $-\operatorname{Art}(X/K)$ replaced by the *conductor exponent* $\varphi(X/K)$. The invariant $\varphi(X/K)$ is the conductor of the $\operatorname{Gal}(\overline{K}/K)$-representation $H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for any $\ell \neq \operatorname{char} k$. It depends only on X and not on any particular integral model for X. The following result of Liu relates $\varphi(X/K)$ to $\operatorname{Art}(\mathcal{X}/\mathcal{O}_K)$ for a proper flat regular $\mathcal{O}_K$-model $\mathcal{X}$ of X:

Proposition 1.8 ([Liu94, Proposition 1])**.** *If $\mathcal{X}/\mathcal{O}_K$ is a proper flat regular model of X, then*

$$-\operatorname{Art}(\mathcal{X}/\mathcal{O}_K) = N - 1 + \varphi(X/K),$$

where N is the number of irreducible components of the (geometric) special fiber $\mathcal{X}_s$ of $\mathcal{X}$.

Applying Proposition 1.8 to the model $\mathcal{X}_f$ in Theorem 1.4 immediately yields the so-called "conductor exponent-discriminant inequality" below, which is Kohl's result when f is monic.

Corollary 1.9 (Conductor exponent-discriminant inequality)**.** *In the situation of Theorem 1.4, we have $\varphi(X/K) \leq (n-1)\nu_K(\operatorname{disc}_n(f))$.*

In Corollary 10.14 we give an easier proof of Corollary 1.9 without using Theorem 1.4.

1.2. **Method of proof.** Theorem 1.4 is fundamentally about bounding the number of irreducible components of the geometric special fiber of a regular model of X. The idea, similar to that of [OS24] in the hyperelliptic case, is to start by considering the normalization $\nu_0\colon \mathcal{X}_0 \to \mathbb{P}^1_{\mathcal{O}_K} =: \mathcal{Y}_0$ in $K(X)$. The singularities on $\mathcal{X}_0$ arise from singularities of the branch locus of ν_0, and the invariant $\operatorname{disc}_n(f)$ measures the "overall badness" of these singularities (see Lemma 2.1 and Proposition 2.4). In the hyperelliptic case, one successively blows up reduced closed points on $\mathbb{P}^1_{\mathcal{O}_K}$, until the cover $\nu_n\colon \mathcal{X}_n \to \mathcal{Y}_n$ obtained by normalizing the blown up model $\mathcal{Y}_n$ in $K(X)$ has a regular branch divisor. This guarantees that the resulting model $\mathcal{X}_n$ of X is regular, and keeping careful track of how much the branch locus singularities are improved by blowing up at every step suffices to yield the conductor-discriminant inequality. Essentially, there is a competition between how quickly one resolves the branch locus singularities and how many irreducible components are added to the special fiber of the model of X, and the bookkeeping can be arranged such that, at every step, the "resolving the branch locus" side wins.

In our situation, where the cover $X \to \mathbb{P}^1_K$ is cyclic of degree n, applying this method runs into several obstacles. Most importantly, for $n > 3$, it is in general *not possible* to find a regular model of $\mathbb{P}^1_K$ whose normalization in $K(X)$ is regular (see [LL99, Proposition 6.6] with $g = 0$ in that proposition). In particular, it is not always possible to start with $\mathbb{P}^1_{\mathcal{O}_K}$ and blow up successive reduced closed points to obtain a model whose normalization in $K(X)$ has regular branch locus. We attempt to solve this problem by blowing up until we obtain a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ such that the branch locus of its normalization $\mathcal{X}$ in $K(X)$ is a *simple normal crossings* divisor, that is, consists of regular irreducible components that meet transversely. It is indeed possible to do this, and in this case, every singularity of $\mathcal{X}$ is a *tame cyclic quotient singularity*, which has an explicit resolution in terms of Hirzebruch–Jung theory.

We play a similar game to that of [OS24], comparing the improvement in the the branch locus singularities to the number of components added to the special fiber of the model of X after every blow up. In the case of general n, quantifying this improvement is significantly more complicated, especially when dealing with a branch locus singularity of multiplicity 2. These calculations are carried out in detail in [Ste26] by the third author, and we quote the main results as Propositions 10.13 and 10.15. In the end, the regular model of X resulting from resolving all the tame cyclic quotient singularities in $\mathcal{X}$ usually satisfies Theorem 1.4, but unfortunately the special fiber sometimes has too many irreducible components.

To resolve this final issue, we do not blow up $\mathbb{P}^1_{\mathcal{O}_K}$ all the way to the point where the branch locus has normal crossings. Instead we stop in certain cases where the branch locus has a multiplicity 2 point whose local neighborhood, when normalized in $K(X)$, yields a *rational double point* (or Du Val singularity). It turns out that applying the well-known resolutions of these singularities from [Lip69] is more efficient than the process of resolving the branch locus to normal crossings and then resolving the resulting tame cyclic quotient singularities, and this is good enough to prove Theorem 1.4.

1.3. **Other results.**

Along the way to our proof of Theorem 1.4, we prove two other results that may be of independent interest.

- If $X \to \mathbb{P}^1_K$ is an $\mathbb{Z}/n$-cover, we give a formula (see Proposition 4.1) for the Swan conductor of X in terms of its defining equation when n is coprime to the residue characteristic. This formula is similar to one of Spencer ([Spe26, Corollary 1.7, Remark 4.3]) for the Swan conductor of a curve X equipped with an *arbitrary* degree n map to $\mathbb{P}^1_K$, although Spencer has the slightly stronger assumption that $\operatorname{char} k > n$. Our technique is different from that of [Spe26], and yields a significantly shorter proof in the cyclic case.
- In Proposition 7.4, we explicitly resolve tame cyclic quotient singularities on arithmetic surfaces in a more general context than that given in, say, [HN16], where the singularity is assumed to arise as a tame cyclic quotient of a local arithmetic surface that is regular *with normal crossings*. We prove that the standard Hirzebruch–Jung-inspired algorithm works without the assumption of normal crossings, and that it in fact gives the *minimal* resolution.

1.4. **Outline of paper.** In §2, we explain why the n-adapted discriminant detects collisions between the branch points of the $\mathbb{Z}/n$-cover on reduction. In §3, we explain why we may assume that the residue field k is algebraically closed throughout without any loss of generality. In §4, we derive a formula for the Swan conductor. In §5, we introduce the *discriminant bonus*, a quantity that is useful when we rephrase our inequalities. In §6, we further reduce to the case where the degree of the defining polynomial f in the defining equation $y^n = f(x)$ for the cover is divisible by n. In §7, we adapt known results about explicit resolution of tame cyclic quotient singularities to our situation. In §8 we derive an inclusion–exclusion formula for the Artin conductor using properties of the compactly supported étale Euler characteristic function. In §9, we rewrite the conductor-discriminant (resp. conductor exponent-discriminant) inequality for X in terms of the positivity of a quantity called the "conductor discriminant contribution" (resp. "conductor exponent-discriminant contribution") associated to multiplicity 2 isolated singularities on particular models of X. This quantity encodes the "resolution versus components" game alluded to in §1.2. Lastly, in §10, we use results from [Ste26] to prove the rephrased inequalities from §9.

Notation and conventions

Throughout, K is a Henselian field with respect to a discrete valuation ν_K, and $\mathcal{O}_K$ is its valuation ring, assumed to be excellent, and k is its residue field, assumed to be perfect. We denote separable and algebraic closures of K by $K^{\mathrm{sep}} \subseteq \overline{K}$. We fix a uniformizer π_K of ν_K and normalize ν_K so that $\nu_K(\pi_K) = 1$.

For a finite separable field extension L/K, we let $\operatorname{disc}(L/K)$ denote the discriminant of the field extension L/K and let $\Delta_{L/K} := \nu_K(\operatorname{disc}(L/K))$.

For an integral K-scheme or $\mathcal{O}_K$-scheme S, we denote the corresponding function field by $K(S)$. If $\mathcal{Y} \to \mathcal{O}_K$ is an arithmetic surface, an irreducible codimension 1 subscheme of $\mathcal{Y}$ is called *vertical* if it lies in a fiber of $\mathcal{Y} \to \mathcal{O}_K$, and *horizontal* otherwise. For two divisors D, D' on a normal arithmetic surface $\mathcal{Y}$, we say $D \leq D'$ or $D' \geq D$ if $D' - D$ is an effective Weil divisor on $\mathcal{Y}$. Let $f \in K(\mathcal{Y})$. We denote the divisor of zeroes of f by $\operatorname{div}_0(f)$. If E is a regular codimension 1 point of $\mathcal{Y}$, we will denote the corresponding discrete valuation on the function field of $\mathcal{Y}$ by ν_E. If P is a closed point on $\mathcal{Y}$, we denote the corresponding local ring by $\mathcal{O}_{\mathcal{Y},P}$ and maximal ideal by $\mathfrak{m}_{\mathcal{Y},P}$. By a singular point of an arithmetic surface $\mathcal{Y}$, we mean a closed point P such that $\operatorname{length} \mathfrak{m}_{\mathcal{Y},P}/\mathfrak{m}_{\mathcal{Y},P}^2 \neq 2$. For a Cartier divisor Δ on an arithmetic surface $\mathcal{Y}$ and a closed point y on Δ, we let $\mu_y(\Delta)$ denote the multiplicity of the point y on Δ, i.e., if f is a defining equation for Δ in $\mathcal{O}_{\mathcal{Y},P}$, then $\mu_y(\Delta) = \max\{r \mid f \in \mathfrak{m}_{\mathcal{Y},P}^r\}$. The branch divisor of a finite morphism of arithmetic surfaces is always considered as a reduced scheme.

Throughout this paper, we will let $X \to \mathbb{P}^1_K$ be a $\mathbb{Z}/n$-cover of $\mathbb{P}^1$ over K of genus $g \geq 1$. Write $\mathbb{P}^1_K = \operatorname{Proj} K[x_0, x_1]$ and let $x := x_1/x_0$. Let $\mathbb{P}^1_{\mathcal{O}_K} := \operatorname{Proj} \mathcal{O}_K[x_0, x_1]$. We assume that X is given by an (affine) equation $y^n = f$, with $f \in \mathcal{O}_K[x]$ that is n-th power free.

On a regular arithmetic surface, the intersection number of two vertical divisors D_1 and D_2 will be written (D_1, D_2).

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2. The geometry of the n-adapted discriminant

In this section, we prove that the n-adapted discriminant $\mathrm{disc}_n(f)$ from Definition 1.1 detects collisions between the components of the branch divisor of the cover $y^n = f(x)$ (Lemma 2.1), and thus serves as a measure of degeneracy for a $\mathbb{Z}/n$-cover of $\mathbb{P}^1_K$ (Proposition 2.4). Recall $\mathbb{P}^1_{\mathcal{O}_K} := \operatorname{Proj} \mathcal{O}_K[x_0, x_1]$ is the standard model $\mathbb{P}^1_K$, that $x := x_1/x_0$, and that X is the curve with affine equation $y^n = f(x)$. Let $\mathcal{X}$ be the normalization of $\mathbb{P}^1_{\mathcal{O}_K}$ in $K(X)$. Let $B \subseteq \mathbb{P}^1_{\mathcal{O}_K}$ be the branch locus of $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ with its reduced induced subscheme structure. Write $B = H + V$ for the decomposition of B into its horizontal and vertical parts. If φ is the structure morphism $H \to \mathcal{O}_K$, let $\mathrm{disc}(H \to \mathcal{O}_K)$ be the closed subscheme $\Delta(\varphi)$ of $\operatorname{Spec} \mathcal{O}_K$ defined in [EH00, Definition V-21].

Lemma 2.1.

(i)

$$\nu_K(\mathrm{disc}_n(f)) = \begin{cases} \mathrm{length}(\mathrm{disc}(H \to \mathcal{O}_K)) & \text{if } V = 0, \text{ and,} \\ \mathrm{length}(\mathrm{disc}(H \to \mathcal{O}_K)) + 2H \cdot V - 2 & \text{if } V \neq 0. \end{cases}$$

In particular $\nu_K(\mathrm{disc}_n(f)) = 0$ *if and only if the map* $B \to \mathcal{O}_K$ *is smooth.*

(ii) *The model* $\mathcal{X}$ *is a smooth model for* X *if and only if* $\nu_K(\mathrm{disc}_n(f)) = 0$.

Proof.

(i) Recall from Definition 1.1 that we defined $\widetilde{f}_n(x_0, x_1) := x_0^d f(x_1/x_0)$, where $d = n\lceil \deg(f)/n \rceil$ and $g = \mathrm{rad}(f_n) \in \mathcal{O}_K[x_0, x_1]/\mathcal{O}_K^\times$. By construction $\mathrm{div}(g) = B$, and further if we write $g = ch$ with $c \in \mathcal{O}_K$ and $h \in \mathcal{O}_K[x_0, x_1]$ such that π_K does not divide h, then $V(c) = V$ and $V(h) = H$. By Definition 1.1 and [GKZ08, Chapter 12, Equation (1.23)],

$$\mathrm{disc}_n(f) = \mathrm{disc}(g) = c^{2(\deg(h)-1)} \,\mathrm{disc}(h),$$

which implies that

$$\nu_K(\mathrm{disc}_n(f)) = \nu_K(\mathrm{disc}(h)) + (2\deg(h) - 2)\nu_K(c).$$

Since $\nu_K(c) = 1$ if $V \neq 0$ and $\nu_K(c) = 0$ if $V = 0$, and since $\deg(h) = \deg(h \mod \pi_K)$ by our assumption that π_K does not divide h, and since $\deg(h \mod \pi_K) = H \cdot V$ by definition, it follows that $(2\deg(h) - 2)\nu_K(c)$ equals $2H \cdot V - 2$ if $V \neq 0$ and is 0 if $V = 0$. Since $\mathrm{disc}(h)$ cuts out the subscheme $\mathrm{disc}(H \to \mathcal{O}_K)$ of $\operatorname{Spec} \mathcal{O}_K$ by [EH00, Corollary V-23], it follows that $\nu_K(\mathrm{disc}(h)) = \mathrm{length}(\mathrm{disc}(H \to \mathcal{O}_K))$. Combining the last two sentences we get the desired equality.

Since X has genus ≥ 1, the cover $X \to \mathbb{P}^1_K$ is branched at at least three points, so when $V \neq 0$ we have $H \cdot V \geq 3$. Combined with the previous paragraph, it follows that $\nu_K(\mathrm{disc}_n(f)) = 0$ precisely when the components of the branch divisor B are pairwise disjoint and $H \to \mathcal{O}_K$ is étale. These conditions are together equivalent to $B \to \mathcal{O}_K$ being étale.

(ii) In light of part (i), we prove that $\mathcal{X} \to \mathcal{O}_K$ is smooth if and only if $B \to \mathcal{O}_K$ is smooth. If $V \neq 0$, then the special fiber of $\mathcal{X}$ is not reduced and thus not smooth. So we may assume $V = 0$ (meaning that $\mathrm{rad}(f)$ is monic), and show that the map $\mathcal{X} \to \mathcal{O}_K$ is smooth if and only if the map $H \to \mathcal{O}_K$ is smooth.

For the "if" direction, since k is algebraically closed by assumption, it follows that the map $H \to \mathcal{O}_K$ is smooth precisely when H is a disjoint union of sections of $\mathbb{P}^1_{\mathcal{O}_K} \to \mathcal{O}_K$, or equivalently that $\operatorname{rad}(f)$ splits into linear factors over K and remains separable when reduced to k. In this case $\mathcal{X} \to \mathcal{O}_K$ is smooth by [SGA1, Exposé XIII, Proposition 5.4] applied to $S = \operatorname{Spec} \mathcal{O}_K$ and X being the localization of $\mathbb{P}^1_{\mathcal{O}_K}$ at each closed point of H.

For the "only if" direction, assume that $\mathcal{X} \to \mathcal{O}_K$ is smooth, let $t \in \mathbb{P}^1_{\mathcal{O}_K}$ be a closed point of H, and let $z \in \mathcal{X}$ be a point above t. Then $\hat{\mathcal{O}}_{\mathbb{P}^1_{\mathcal{O}_K},t} \cong (\hat{\mathcal{O}}_{\mathcal{X},z})^{I_z}$, where $I_z \leq \mathbb{Z}/n$ is the inertia group at z, and both rings are power series rings in one variable over $\mathcal{O}_K$. Let d_k be the degree of the ramification divisor of the special fiber of $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ at z, and let d_K be the sum of degrees of the ramification divisor of the generic fiber $X \to \mathbb{P}^1_K$ at all points specializing to z. Then $d_k = d_K$ by [GM98, I, Claim 3.2]. A geometric point of $\mathbb{P}^1_K$ with ramification index e for the cover contributes $|I_z|(1 - 1/e) \geq |I_z|/2$ to the sum d_K. We also have that $d_k = |I_z| - 1$. Combining the last three sentences, we conclude that there is only one such geometric point, which means that $H \to \mathcal{O}_K$ is locally of degree 1 at t, or equivalently a local isomorphism, and thus $H \to \mathcal{O}_K$ is smooth at t. $\square$

Remark 2.2. Since $V = \mathbb{P}^1_k$ when $V \neq 0$, using the convention that $p_a(\emptyset) = 1$[2], Lemma 2.1 above can be restated more uniformly as

$$\nu_K(\operatorname{disc}_n(f)) = \operatorname{length}(\operatorname{disc}(H \to \mathcal{O}_K)) + 2H \cdot V + 2p_a(V) - 2. \tag{2.3}$$

Thus, we can break $\nu_K(\operatorname{disc}_n(f))$ into three parts: The first part $\operatorname{disc}(H \to \mathcal{O}_K)$, accounts for collisions between the horizontal components of B, the second part $2H \cdot V$, accounts for collisions (if any) between the unique vertical component of B with the horizontal components of B, and the third part accounts for collisions between the vertical components of B. In forthcoming work [OS26], the first two authors use Equation (2.3) to define the discriminant of the branch divisor on an arbitrary regular arithmetic surface $\mathcal{Y}$ in place of $\mathbb{P}^1_{\mathcal{O}_K}$ to study conductor–discriminant inequalities in a more general setting. In the case of general $\mathcal{Y}$, the vertical branch divisor V may have more than one irreducible component and the appropriate generalization of $\nu_K(\operatorname{disc}_n(f))$ would need to account for collisions between different vertical branch components.

Proposition 2.4. *Let X be a nice curve with a smooth proper $\mathcal{O}_K$-model $\mathcal{X}$. Assume that X admits a $\mathbb{Z}/n$-action such that the quotient is isomorphic to $\mathbb{P}^1_K$. Assume that the genus g of X is at least 1. Then the $\mathbb{Z}/n$-action extends to $\mathcal{X}$, and there is an isomorphism $\mathcal{X}/(\mathbb{Z}/n) \cong \mathbb{P}^1_{\mathcal{O}_K}$. Moreover, there is an affine equation for X of the form $y^n = f(x)$ with $f(x) \in \mathcal{O}_K[x]$ that is nth power free and satisfies $\nu_K(\operatorname{disc}_n(f)) = 0$.*

Proof. Since $g \geq 1$ by assumption, the minimal proper regular model of X is unique up to unique isomorphism (cf. [Liu06, Definition 3.31, Proposition 3.36 (2)]) and since the smooth model $\mathcal{X}$ is the minimal model, it follows that the $\mathbb{Z}/n$-action extends to $\mathcal{X}$. If $\mathcal{Y} := \mathcal{X}/(\mathbb{Z}/n)$, then [Liu06, Proposition 10.3.48(a)] implies that $\mathcal{Y}$ is smooth, and thus isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$.

Since $n \nmid \operatorname{char}(k)$ and since K is Henselian, it follows that K has all nth roots of unity. The final claim follows from Lemma 2.1 (ii) by choosing explicit coordinates on $\mathbb{P}^1_{\mathcal{O}_K}$ and applying Kummer theory. $\square$

[2]This convention is reasonable if we think of $\dim(\emptyset) = -1$ and $\chi(\emptyset, \mathcal{O}_\emptyset) = 0$.

3. Reduction to the case of algebraically closed residue field

In this section, we explain why for Theorem 1.4 and Corollary 1.9, we may assume that the residue field k is *algebraically closed* rather than just perfect for the rest of the paper without any loss of generality. Firstly, replacing K with an unramified extension does not affect $\nu_K(\mathrm{disc}_n(f))$. Secondly, base changes to unramified extensions of K do not affect the conductor exponent $\varphi(X/K)$. Thirdly, (normalized) base changes of any regular model $\mathcal{X}/\mathcal{O}_K$ to unramified extensions of $\mathcal{O}_K$ preserve regularity and commute with reduction to the special fiber. In particular, to construct a regular model $\mathcal{X}_f$ as in Theorem 1.4, it suffices to construct such a model $\mathcal{X}_f^{ur}$ over the ring of integers $\mathcal{O}_K^{ur}$ of the maximal unramified extension K^{ur} of K, then let $\mathcal{X}_f := \mathcal{X}_f^{ur}/\operatorname{Gal}(K^{ur}/K)$; observe that the quantities on the right hand side of the equation in Proposition 1.8 are the same before and after taking the quotient. Thus $\operatorname{Art}(\mathcal{X}_f/\mathcal{O}_K) = \operatorname{Art}(\mathcal{X}_f^{ur}/\mathcal{O}_K^{ur})$. So the inequality in Theorem 1.4 holds for $\mathcal{X}_f$ so long as it holds for $\mathcal{X}_f^{ur}$. So from now on we assume that k is algebraically closed.

Since K is Henselian and k is algebraically closed with $\operatorname{char} k \nmid n$, all units in $\mathcal{O}_K$ are nth powers, and the group $K^\times/(K^\times)^n$ has order n and is generated by the image of π_K. Since multiplying f by nth powers does not change the isomorphism class of X, without loss of generality we may assume that the maximum power of π_K dividing f in $\mathcal{O}_K[x]$ is π_K^b for some $b \in \{0, \dots, n-1\}$.

Assumption 3.1. Assume $\operatorname{char} k \nmid n$ and k is algebraically closed. We write the irreducible factorization of f as

$$\pi_K^b f_1^{d_1} \dots f_r^{d_r}, \tag{3.2}$$

where $b \in \{0, \dots, n-1\}$, the $d_i \in \{1, \dots, n-1\}$, and the $f_i \in \mathcal{O}_K[x]$ are distinct irreducible separable polynomials. For each $1 \le i \le r$, we pick a root α_i of f_i in $\overline{K}$ and set $K_i = K(\alpha_i)$.

4. A computable formula for the Swan conductor

Proposition 4.1. *Let f, K_i be as in Assumption 3.1, and let X/K be the smooth projective curve with affine equation $y^n = f(x)$. The Swan conductor $\delta(X/K)$ for the curve X equals*

$$\sum_{i=1}^{r} (n - \gcd(n, d_i))(\Delta_{K_i/K} - \deg(f_i) + 1).$$

Proof. Let L/K be a finite Galois extension over which f splits, and let G_j be the jth higher ramification group of $G := \operatorname{Gal}(L/K)$ for the lower numbering. Let B_i be the set of roots of f_i, and let $B = \bigsqcup_i B_i$. Since the f_i are irreducible, G acts on each B_i. Denote the special fiber of the quasi-stable model $\mathcal{X}^{qs}$ of X constructed in [BW17, §4] by $\overline{X}^{qs}$. The model $\mathcal{X}^{qs}$ is a semi-stable model for which all ramification points of $X \to \mathbb{P}^1$ specialize to distinct smooth points. By construction of $\mathcal{X}^{qs}$, it follows that $\Gamma := \operatorname{Gal}(X/\mathbb{P}^1)$ acts on $\overline{X}^{qs}$ and we write $\overline{Y}^{qs} := \overline{X}^{qs}/\Gamma$. The group G acts on $\overline{X}^{qs}$ by k-automorphisms and commutes with the action of Γ, so its action descends to $\overline{Y}^{qs}$. Since arithmetic genus is constant in flat families, we have $p_a(\overline{X}^{qs}) = g(X)$ and $p_a(\overline{Y}^{qs}) = 0$.

By [BW17, Theorem 2.13], we have the formula

$$\delta(X/K) = \sum_{j=1}^{\infty} \frac{|G_j|}{|G_0|} \cdot (2p_a(\overline{X}^{qs}) - 2p_a(\overline{X}^{qs}/G_j)). \tag{4.2}$$

For $j \geq 1$, we claim that

$$2p_a(\overline{X}^{qs}) - 2p_a(\overline{X}^{qs}/G_j) = \sum_{i=1}^{r}(n - \gcd(n, d_i))(|B_i| - |B_i/G_j|). \tag{4.3}$$

To prove (4.3), note that since the actions of Γ and G commute, the action of Γ descends to $\overline{Y}/G_j$ for any j. Since the group $|G_j|$ is a p-group for all $j \geq 1$, it cannot permute a Γ-orbit of $\overline{X}^{qs}$ non-trivially. Consequently, $\overline{X}^{qs}/G_j \to \overline{Y}^{qs}/G_j$ is a map of semistable curves that is generically separable of degree n above each irreducible component of the target. By construction, the branch points of $X \to \mathbb{P}^1$ specialize to distinct smooth points on $\overline{Y}^{qs}$. Identifying B with the branch locus of $X \to \mathbb{P}^1$ away from ∞, we can also identify the smooth points of $\overline{Y}^{qs}$ in the branch locus of $\overline{X}^{qs} \to \overline{Y}^{qs}$ away from the specialization of ∞ with B via specialization. Carrying on, the smooth points of $\overline{Y}^{qs}/G_j$ in the branch locus of $\overline{X}^{qs}/G_j \to \overline{Y}^{qs}/G_j$ away from the specialization of ∞ correspond to the elements of $\bigsqcup_i B_i/G_j$. Furthermore, the ramification index above each point of B_i/G_j is $n/\gcd(n, d_i)$. Since $p_a(\overline{Y}^{qs}/G_j) = 0$, the Riemann-Hurwitz formula for semistable curves (see, e.g., [Obu09, Proposition 2.15]) shows that

$$2p_a(\overline{X}^{qs}) - 2 = -2n + \sum_{i=1}^{r}(n - \gcd(n, d_i))|B_i| + |R_\infty| \tag{4.4}$$

and

$$2p_a(\overline{X}^{qs}/G_j) - 2 = -2n + \sum_{i=1}^{r}(n - \gcd(n, d_i))|B_i/G_j| + |R_\infty|, \tag{4.5}$$

where R_∞ is the ramification divisor above ∞. Combining (4.4) and (4.5) proves the claim.

From (4.2) and (4.3), we obtain

$$\begin{aligned} \delta(X/K) &= \sum_{j=1}^{\infty} \frac{|G_j|}{|G_0|} \sum_{i=1}^{r}(n - \gcd(n, d_i))(|B_i| - |B_i/G_j|) \\ &= \sum_{i=1}^{r}(n - \gcd(n, d_i)) \sum_{j=1}^{\infty} \frac{|G_j|}{|G_0|}(|B_i| - |B_i/G_j|). \end{aligned} \tag{4.6}$$

If V_i is a vector space with G_j-stable basis B_i, then $\dim(V_i^{G_j}) = |B_i/G_j|$. So for each i, the sum over j on the right-hand side of (4.6) is, by definition, the Swan conductor of the permutation representation ρ of G on the roots of f_i. Now, ρ is induced from the trivial representation of $H_i := \mathrm{Gal}(L/K_i)$. By a variant of the conductor-discriminant formula (see [Ser79, p. 101, Corollary to Proposition 4]), the Artin conductor of ρ is $\Delta_{K_i/K}$. Since K_i/K is totally ramified, we have $G_0 = G$ and $V_i^{G_0}$ is generated by $(1, 1, \ldots, 1)$. G_0 acts on B_i with one fixed point. Thus $\dim(V_i) - \dim(V_i^{G_0}) = \deg(f_i) - 1$, so the Swan conductor of ρ is $\Delta_{K_i/K} - \deg(f_i) + 1$. Plugging this into (4.6) proves the proposition. □

5. The discriminant bonus

Let $\mathrm{disc}_n(f)$ be the n-adapted discriminant of f as in Definition 1.1, and let K_i be as in Assumption 3.1.

Definition 5.1. The *discriminant bonus of f over K*, written $\mathrm{db}_K(f)$, is the quantity $\nu_K(\mathrm{disc}_n(f)) - \sum_{i=1}^r \Delta_{K_i/K}$.

Remark 5.2. This generalizes the definition of the discriminant bonus given in [OS24, Definition 2.4] from $n = 2$ to arbitrary n.

Remark 5.3. If $f \in \mathcal{O}_K[x]$ is irreducible and monic, α is a root of f, and $L = K(\alpha)$, then $\mathrm{db}_K(f)$ is twice the length of $\mathcal{O}_L/\mathcal{O}_K[\alpha]$ as an $\mathcal{O}_K$-module. We do not use this, so we omit the proof.

Remark 5.4. When $n \mid \deg(f)$, the product formula for the discriminant (cf. [GKZ08, Chapter 12, Equation (1.23)]) implies that

$$\nu_K(\mathrm{disc}_n(f)) = \nu_K(\mathrm{disc}(\mathrm{rad}(f)) = \nu_K(\mathrm{disc}(\mathrm{rad}(f/\pi_K^b))) + \begin{cases} 0 & b = 0 \\ 2\deg(\mathrm{rad}(f)) - 2 & b \geq 1 \end{cases},$$

and hence by Definition 1.1, it follows that

$$\mathrm{db}_K(f) = \mathrm{db}_K(f/\pi_K^b) + \begin{cases} 0 & b = 0 \\ 2\deg(\mathrm{rad}(f)) - 2 & b \geq 1 \end{cases}.$$

6. Reduction to the case where $\deg f$ is divisible by n

Changing coordinates on $\mathbb{P}^1_K$ using an element of $\mathrm{GL}_2(\mathcal{O}_K)$ does not change $\nu_K(\mathrm{disc}_n(f))$ (see, e.g., [GKZ08, Ch. 12, (1.26)]). Since k is algebraically closed, we may assume by such a change of coordinates that no component of the branch divisor of $X \to \mathbb{P}^1_K$ specializes to ∞ (see, e.g., [Sri15, Section 1.3]). In other words, we may assume that all roots of f are integral over $\mathcal{O}_K$, and that $n \mid \deg(f)$. Thus, for the remainder of the paper, we make the following assumption:

Assumption 6.1. Assume f, f_i, n are as in Assumption 3.1. Additionally assume that the f_i are monic, and that $n \mid \deg(f)$.

The argument above proves the following proposition.

Proposition 6.2. *If the conductor-discriminant inequality holds for all f satisfying Assumption 6.1, then it holds for all $f \in \mathcal{O}_K[x]$.*

7. Resolution of tame cyclic quotient singularities

In this section, we generalize several results about the resolution of tame cyclic quotient singularities from [HN16] and [Ste18], dropping the additional "stable-setting hypothesis" (Definition 7.1) made in [HN16] and [Ste18].

Definition 7.1. Let $\nu, r \in \mathbb{Z}$ such that $1 \leq r < \nu$, $\mathrm{char}(k) \nmid \nu$, and $\gcd(\nu, r) = 1$. Fix a primitive ν-th root of unity $\zeta \in \mathcal{O}_K$ and a generator $\sigma \in \mathbb{Z}/\nu\mathbb{Z}$. Consider the $\mathbb{Z}/\nu\mathbb{Z}$-action on $\mathcal{O}_K[[t_1, t_2]]$ which is trivial on $\mathcal{O}_K$ and satisfies $\sigma(t_1) = \zeta t_1$ and $\sigma(t_2) = \zeta^r t_2$. Let $\varphi \in \mathcal{O}_K[[t_1, t_2]]$ be a power series fixed under this $\mathbb{Z}/\nu\mathbb{Z}$-action, and consider the induced $\mathbb{Z}/\nu\mathbb{Z}$-action on $A := \mathcal{O}_K[[t_1, t_2]]/(\pi_K - \varphi)$.

Assume A is flat over $\mathcal{O}_K$, that is, $\pi_K \nmid \varphi$. An $\mathcal{O}_K$-scheme $\mathcal{X}$ isomorphic to Spec $A/(\mathbb{Z}/\nu)$ is called a *tame cyclic quotient singularity of type $\frac{1}{\nu}(1, r)$*. If moreover we can take $A = \mathcal{O}_K[[t_1, t_2]]/(\pi_K - t_1^{a_1} t_2^{a_2})$ for some nonnegative integers a_1, a_2 with $a_1 \geq 1$, then $\mathcal{X}$ is said to *admit a stable setting* (see [Ste18, Definition 4.1.5, Theorem 4.1.6, and Theorem 4.3.4]).

Definition 7.2. Let $\nu, r \in \mathbb{Z}$ be as in Definition 7.1. Let $r_0 = \nu, r_1 = r$, and while $r_i \neq 0$, let r_{i+1} be the least residue of $-r_{i-1}$ mod r_i. The largest index i for which r_i is defined and nonzero is called the *Hirzebruch–Jung length of the pair* (ν, r) and is denoted $L(\nu, r)$.

Remark 7.3. Observe that $L(\nu, r) \leq \nu - 1$ in Definition 7.2 since the r_i form a decreasing sequence with $r_1 = r < \nu$.

Setting $b_i = \lceil \frac{r_{i-1}}{r_i} \rceil$, we recover the Hirzebruch–Jung (negative) continued fraction expansion for the fraction ν/r as:

$$b_1 - \cfrac{1}{b_2 - \frac{1}{\dots - \frac{1}{b_{L(\nu,r)}}}}.$$

We write $\nu/r = [b_1, b_2, \dots, b_L]_{HJ}$ with $L = L(\nu, r)$, see [HN16, Definition 4.4.2.4].

Proposition 7.4. *Let $\mathcal{X}$ be a tame cyclic quotient singularity of type $\frac{1}{\nu}(1, r)$ with closed point x, let $L = L(\nu, r)$, and suppose $\nu/r = [b_1, \dots, b_L]_{HJ}$.*

(i) *The reduced exceptional divisor of the minimal resolution $\pi : \mathcal{X}' \to \mathcal{X}$ is a chain $E_1 - E_2 - \dots - E_L$ of L components isomorphic to $\mathbb{P}^1_k$, i.e., for each $2 \leq i \leq L-1$, the component E_i intersects E_{i-1} and E_{i+1} each transversely in a unique point, and*

(ii) *For $1 \leq i \leq L$, the self-intersection number of E_i is $-b_i \leq -2$.*

Proof. If $\mathcal{X}$ admits a stable setting, then this is [HN16, Proposition 4.4.2.5][3]. We now prove the result in the non-stable setting case. The proof is still parallel to that of [HN16, Proposition 4.4.2.5] and we adopt its notation (our ν plays the role of n in [HN16]). Let φ be as in Definition 7.1. The proof of [HN16, Lemma 4.4.2.2] goes through exactly, replacing $wt_1^{m_1}t_2^{m_2}$ in [HN16] with our φ. As in the proof of [HN16, Proposition 4.4.2.5], set $u_0 = t_1^\nu$, $v_0 = t_1^{-r}t_2$, and define inductively $u_i = v_{i-1}^{-1}$ and $v_i = u_{i-1}v_{i-1}^{b_i}$ for $i \geq 1$. Letting S be the set of monomials in t_1 and t_2 invariant under the $\mathbb{Z}/\nu\mathbb{Z}$-action ([HN16, Lemma 4.4.2.2]), one defines

$$B_{\sigma_i} = \mathcal{O}_K[[S]][u_i, v_i]/(\pi_K - \varphi_i) \subseteq \mathcal{O}_K[[u_i, v_i]]/(\pi_K - \varphi_i)$$

as in the proof of [HN16, Proposition 4.4.2.5], where φ_i is just φ written in terms of the variables u_i and v_i. Note that each monomial of the form $wt_1^{m_1}t_2^{m_2}$ in φ (with $w \in \mathcal{O}_K$) can be written as $wu_i^{\mu_{i+1}}v_i^{\mu_i}$ as part of φ_i, where the μ_i are integers defined as in the proof of [HN16, Proposition 4.4.2.5] applied to the case $\varphi = wt_1^{m_1}t_2^{m_2}$. In particular, $\mu_0 \geq 0$ and $\mu_i > 0$ for all $i \geq 1$ as in [HN16], which means that $\varphi_i \in \mathcal{O}_K[[S]][u_i, v_i]$ is divisible by u_i for $0 \leq i \leq L-1$ and by v_i for $1 \leq i \leq L$.

Let $\mathcal{X}'$ be obtained from gluing together the Spec B_{σ_i} by identifying u_i with v_{i-1}^{-1} and v_i with $u_{i-1}v_{i-1}^{b_i}$ for $i \geq 1$. The proof that B_{σ_i} is a regular ring and that u_i, v_i form a regular system of parameters now goes through verbatim as in the proof of [HN16, Proposition 4.4.2.5], as does the proof that the exceptional divisors $E_1, \dots, E_L$ are all isomorphic to $\mathbb{P}^1_k$. The divisor E_i is fully contained in Spec $B_{\sigma_i} \cup$ Spec $B_{\sigma_{i-1}}$, with $E_i \cap \text{Spec } B_{\sigma_i} = \text{div}(v_i)$ and $E_i \cap \text{Spec } B_{\sigma_{i-1}} = \text{div}(u_{i-1})$. The point $v_{i-1} = 0 \in E_i \cap \text{Spec } B_{\sigma_{i-1}}$ is the unique point of E_i not contained in Spec B_{σ_i}. The fact that E_i meets E_{i+1} transversely for $1 \leq i \leq L-1$ also follows as in loc. cit. by ignoring the factors of φ_i other than powers of u_i and v_i. So

[3]In [HN16], it is assumed that char k does not divide a_1 or a_2 (called m_1 and m_2 there), but this is only used to define the number r, and is not used anywhere else in the proof. Since we define r explicitly in our proposition, this assumption is unnecessary.

$\mathcal{X}' \to \mathcal{X}$ is a resolution of singularities whose exceptional divisor is a chain of transversely meeting $\mathbb{P}^1_k$'s. It remains to show it is the *minimal* resolution, which we do by showing that $(E_i, E_i) = -b_i \leq -2$. This will prove (i) and (ii) together.

Let $\mathcal{X}'_s$ be the special fiber of $\mathcal{X}'$. For $1 \leq i \leq L$, we begin by calculating the multiplicity of E_i in $\mathcal{X}'_s$. Let $\overline{\varphi}_i \in k[[S]][u_i, v_i] \subseteq k[[u_i, v_i]]$ be the reduction of φ_i modulo π_K. By Definition 7.1, $\overline{\varphi}_i$ is nonzero. Let $wu_i^c v_i^d$ be the monomial term of $\overline{\varphi}_i \in k[[u_i, v_i]]$ for which d is minimal, and for which c is maximal given this minimal d.[4] Then there exists $\alpha \in k[u_i]$ with degree less than c, as well as $\beta \in k[[S]][u_i, v_i]$ divisible by v_i such that

$$\overline{\varphi}_i = v_i^d(wu_i^c + \alpha + \beta).$$

Since $\operatorname{div}(v_i) = E_i \cap \operatorname{Spec} B_{\sigma_i}$, we have that the multiplicity of E_i in the special fiber is d.

Since $(E_i, \mathcal{X}'_s) = 0$, we have

$$(E_i, E_i) = -\frac{(E_i, \mathcal{X}'_s - dE_i)}{d}.$$

We compute $(E_i, \mathcal{X}'_s - dE_i)$ as a sum of local intersection numbers at closed points of E_i. First, we observe that the contribution coming from closed points in $E_i \cap \operatorname{Spec} B_{\sigma_i}$ is the degree of $wu_i^c + \alpha + \beta$ as a polynomial in u_i, after setting $v_i = 0$. This equals c. It remains to compute the contribution coming from the unique closed point of E_i not in $\operatorname{Spec} B_{\sigma_i}$. As was mentioned above, this is the point $u_{i-1} = v_{i-1} = 0$ in $\operatorname{Spec} B_{\sigma_{i-1}}$.

Since $u_i = v_{i-1}^{-1}$ and $v_i = u_{i-1}v_{i-1}^{b_i}$, we have $wu_i^c v_i^d = wu_{i-1}^{c'} v_{i-1}^{d'}$, with $c' = d$ and $d' = db_i - c$. Note that c' is minimal among the u_{i-1}-exponents of the terms of $\overline{\varphi}_{i-1}$ and d' is minimal among the v_{i-1}-exponents given this minimal c'. So there exist $\gamma \in k[[v_{i-1}]]$ divisible by $v_{i-1}^{d'+1}$ and $\delta \in k[[S]][u_{i-1}, v_{i-1}]$ divisible by u_{i-1} such that

$$\overline{\varphi}_{i-1} = u_{i-1}^{c'}(wv_{i-1}^{d'} + \gamma + \delta).$$

So the contribution to $(E_i, \mathcal{X}'_s - dE_i)$ coming from this point is number of times v_{i-1} divides $(wv_{i-1}^{d'} + \gamma + \delta)$, after setting $u_{i-1} = 0$. This equals d'.

Summing up, we have $(E_i, \mathcal{X}'_s - dE_i) = c + d'$, which means that

$$(E_i, E_i) = -\frac{c + d'}{d} = -\frac{c + db_i - c}{d} = -b_i.$$

□

The following lemma is probably well-known, but we include a proof for lack of a suitable reference. Recall that a simple normal crossings divisor B on an arithmetic surface $\mathcal{Y}$ is a divisor all of whose irreducible components are regular, and such that at most two irreducible components of B intersect at any point and all such intersections are transverse. Similarly, a simple normal crossings point of B is a point $y \in \operatorname{Supp}(B)$ that is a regular point of B or is contained in exactly two irreducible components of B, which are regular and meet transversely at y.

Lemma 7.5. *Every regular model of $\mathbb{P}^1_K$ has simple normal crossings, and the dual graph of its special fiber is a tree.*

[4]This maximum is attained since u_i, being dual to $e_i \in \mathbb{Q}_{>0} \times \mathbb{Q}_{>0}$ in the proof of [HN16, Proposition 4.4.2.5], has a negative power of either t_1 or t_2 (in fact, t_2), so c cannot be arbitrarily high given a fixed d.

Proof. Let $\mathcal{Y}$ be any regular model of $\mathbb{P}^1_K$ and let $\mathcal{Y}_s$ be the special fiber. Pick a K-rational point on $\mathbb{P}^1_K$; it specializes to a multiplicity 1 component of $\mathcal{Y}_s$, say $\overline{V}$. By [OW20, Lemma 7.1], the contraction of all components of $\mathcal{Y}_s$ other than $\overline{V}$ has $\mathbb{P}^1_k$ as its special fiber, so is a smooth model $\mathcal{Y}^{sm}$ of $\mathbb{P}^1_K$. By [Liu06, Theorem 9.2.2], the contraction morphism $\mathcal{Y} \to \mathcal{Y}^{sm}$ is made up of a finite sequence of blowups along closed points. Since the dual graph of the special fiber of $\mathcal{Y}^{sm}$ is a tree, the same holds for the dual graph of the special fiber of $\mathcal{Y}$. Furthermore, [Liu06, Lemma 9.2.31] shows that $\mathcal{Y}$ has simple normal crossings. $\square$

Proposition 7.6. *Let $\mathcal{Y}$ be regular model of $\mathbb{P}^1_K$, and let D be the branch divisor for the normalization $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{Y}$ in $K(X)$. Assume that the branch divisor D is a simple normal crossings divisor.*

(i) *Every non-regular point of $\mathcal{X}$ lies over the intersection point of two irreducible components of D.*

(ii) *Every closed point of $\mathcal{X}$ where the special fiber is not a simple normal crossings divisor lies over a horizontal irreducible component of D.*

(iii) *Let $y \in \mathcal{Y}$ be the intersection point of two irreducible components D_1, D_2 of D. For $i = 1, 2$, let $\nu_i := \mathrm{ord}_{D_i}(f), e_i := \frac{n}{\gcd(n,\nu_i)}$ and let κ be the least residue mod $\gcd(e_1, e_2)$ of $-\left(\frac{\nu_1}{\gcd(\nu_1,n)}\right)^{-1} \frac{\nu_2}{\gcd(\nu_2,n)}$. There are $\frac{n}{\mathrm{lcm}(e_1,e_2)}$ points of $\mathcal{X}$ lying over y, and for such a point $x \in \mathcal{X}$, the scheme* Spec $\hat{\mathcal{O}}_{\mathcal{X},x}$ *is a tame cyclic quotient singularity of type*

$$\frac{1}{\gcd(e_1, e_2)}(1, \kappa).$$

(iv) *Keep the notation of (iii) and suppose D_1, D_2 are contained in $\mathcal{Y}_s$. If $x \in \mathcal{X}$ lies over y, then the tame cyclic quotient singularity* Spec $\hat{\mathcal{O}}_{\mathcal{X},x}$ *admits a stable setting.*

Proof. Part (i) follows from [KW25, Theorem 1.5], since the only singular points of D are those lying on two or more components. Part (ii) follows from [OS25, Proposition 2.8(i)].

To prove the first assertion of part (iii), let A_1 and A_2 be irreducible components of $\gamma^{-1}(D_1)$ and $\gamma^{-1}(D_2)$, respectively, passing through a point x above y. By the definition of e_1 and e_2, the stabilizer of x in $\mathbb{Z}/n$ contains the group H generated by $\mathbb{Z}/e_1$ and $\mathbb{Z}/e_2$ in $\mathbb{Z}/n$. Since H has order $\mathrm{lcm}(e_1, e_2)$, it suffices to show that y is not a branch point of $p\colon \mathcal{X}/H =: \mathcal{Z} \to \mathcal{Y}$. Since $\mathcal{Y}$ is regular and $\mathcal{X}$ is normal, this follows by purity of the branch locus (see, e.g., [Sza09, Theorem 5.2.13]).

We now prove the second assertion of part (iii). Since $p\colon \mathcal{Z} \to \mathcal{Y}$ is étale over y, we can replace y with any point z of $p^{-1}(y)$ and the D_i with irreducible components of the $p^{-1}(D_i)$ passing through z without changing the e_i. So we may assume $\mathcal{Z} = \mathcal{Y}$, or in other words, that $\mathrm{lcm}(e_1, e_2) = n$.

Since $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is a regular local ring, the height one prime ideals corresponding to the restriction of D_1 and D_2 to $\hat{\mathcal{O}}_{\mathcal{Y},y}$ are principal. Let s_1 and s_2 be their respective generators. By Hensel's lemma, all units in $\hat{\mathcal{O}}_{\mathcal{Y},y}$ are nth powers, so by Kummer theory, the extension $\mathrm{Frac}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \hookrightarrow \mathrm{Frac}(\hat{\mathcal{O}}_{\mathcal{X},x})$ is given by an equation of the form

$$s^n = s_1^{\nu_1} s_2^{\nu_2}. \tag{7.7}$$

In fact, after replacing s with an appropriate prime-to-n power and multiplying it by appropriate powers of s_1 and s_2, we may further assume that $\nu_1 = n/e_1$ and $\nu_2 \in \{0, \dots, n-1\}$, with ν_2 divisible by n/e_2. On the other hand, since $\mathcal{Y}$ is regular with normal crossings by Lemma 7.5, we can write $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong \mathcal{O}_K[[u_1, u_2]]/(\pi_K - \beta u_1^a u_2^b)$ for some $a, b \in \mathbb{Z}_{\geq 0}$ and $\beta \in \mathcal{O}_K[[u_1, u_2]]^\times$. Since s_1 and s_2 generate the maximal ideal of $\hat{\mathcal{O}}_{\mathcal{Y},y}$, there exist power series α_1, resp. α_2 over $\mathcal{O}_K$ such that $\alpha_1(s_1, s_2) = u_1$ and $\alpha_2(s_1, s_2) = u_2$. Consider the ring

$$A := \hat{\mathcal{O}}_{\mathcal{Y},y}[t_1, t_2]/(t_1^{e_1} - s_1, t_2^{e_2} - s_2) \cong \mathcal{O}_K[[t_1, t_2]]/(\pi_K - \beta(t_1^{e_1}, t_2^{e_2})\alpha_1(t_1^{e_1}, t_2^{e_2})^a \alpha_2(t_1^{e_1}, t_2^{e_2})^b).$$

The maximal ideal of A is generated by t_1 and t_2, so A is regular. Furthermore, $t_1^n t_2^{\nu_2 e_2} = s_1^{\nu_1} s_2^{\nu_2} = s^n$ by (7.7). So, up to multiplication by an nth root of unity, we have

$$s = t_1 t_2^{\nu_2 e_2/n}. \tag{7.8}$$

Let ζ be a primitive $(e_1 e_2/n)$-th root of unity and consider the $\mathbb{Z}/(e_1 e_2/n)$-action on A fixing $\mathcal{O}_K$ given by $\sigma(t_1) = \zeta^{-\nu_2 e_2/n} t_1$ and $\sigma(t_2) = \zeta t_2$. By (7.8), σ fixes s, so by the normality of $A^{\langle\sigma\rangle}$ and $\hat{\mathcal{O}}_{\mathcal{X},x}$ we have $A^{\langle\sigma\rangle} = \hat{\mathcal{O}}_{\mathcal{X},x}$. Thus (switching the roles of t_1 and t_2), Spec $\hat{\mathcal{O}}_{X,x}$ is a tame cyclic quotient of type

$$\frac{1}{e_1 e_2/n}(1, -\nu_2 e_2/n).$$

Since $\operatorname{lcm}(e_1, e_2) = n$, we have $e_1 e_2/n = \gcd(e_1, e_2)$, so the type is $\frac{1}{\gcd(e_1,e_2)}(1, -\nu_2 e_2/n)$. Since $\gcd(\nu_2, n) = n/e_2$ and $\nu_1 = n/e_1$, the proof of part (iii) is complete.

Lastly, in the situation of part (iv), we have without loss of generality that $u_1 = s_1$ and $u_2 = s_2$, so $A \cong \mathcal{O}_K[[t_1, t_2]]/(\pi_K - t_1^{e_1 a} t_2^{e_2 b})$, implying that $\hat{\mathcal{O}}_{\mathcal{X},x}$ admits a stable setting. $\square$

Let y, e_1, e_2, κ be as in Proposition 7.6 (iii). In light of Proposition 7.6 (iii), we make the following definition.

Definition 7.9. The *Hirzebruch–Jung length of* y is the nonnegative integer

$$\mathrm{HJL}(y) = L(\gcd(e_1, e_2), \kappa)$$

where L is the Hirzebruch–Jung length of a pair of integers from Definition 7.2.

Remark 7.10. Keep the notation of Definition 7.9. By Remark 7.3, we have

$$\mathrm{HJL}(y) \leq \gcd(e_1, e_2) - 1.$$

Furthermore, if $x \in \mathcal{X}$ lies above $y \in \mathcal{Y}$, then by Proposition 7.4, the Hirzebruch–Jung length of y is the number of irreducible components in the exceptional divisor of the minimal resolution of Spec $\hat{\mathcal{O}}_{\mathcal{X},x}$. By [OW20, Proposition 3.5], the same is true for the minimal resolution of $\mathcal{X}$ in x.

8. An inclusion–exclusion formula for the Artin conductor

In this section, we first recall the definition of the Artin conductor. Note that the definition works not only for regular models, but also for normal models. Then after recalling some properties of the compactly supported Euler characteristic function in Properties 8.2, we use these to derive a formula for the Artin conductor for certain special normal models (Proposition 8.6 and Proposition 8.18).

Let X be a smooth, projective, geometrically integral curve of genus $g \geq 1$ defined over K. Let $\mathcal{X}$ be a proper, flat, *normal* $\mathcal{O}_K$-scheme with generic fiber X.

Definition 8.1. The *Artin conductor* associated to the model $\mathcal{X}$ is defined by

$$\mathrm{Art}'(\mathcal{X}/\mathcal{O}_K) := \chi(\mathcal{X}_{\overline{K}}) - \chi(\mathcal{X}_k) - \delta(X/K),$$

where χ is the Euler characteristic for the ℓ-adic cohomology and $\delta(X/K)$ is the Swan conductor associated to the ℓ-adic representation $\mathrm{Gal}(\overline{K}/K) \to \mathrm{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(\mathcal{X}_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$). Let

$$\mathrm{Art}^+(\mathcal{X}/\mathcal{O}_K) := -\mathrm{Art}'(\mathcal{X}/\mathcal{O}_K).$$

The compactly supported Euler characteristic. Let F be an algebraically closed field (in what follows, F will be $\overline{K}$ or k). Let $\mathcal{V}_F$ denote the set of isomorphism classes of separated F-schemes of finite type. Let ℓ be a prime such that $\ell \neq \mathrm{char}(F)$. For any F-variety X and integer $i \geq 0$, let $H^i_c(X, \mathbb{Q}_\ell)$ denote the i^{th} ℓ-adic étale cohomology group with compact support. The compactly supported Euler characteristic is the function

$$\begin{aligned} \chi \colon \mathcal{V}_F &\to \mathbb{Z} \\ X &\mapsto \sum_{i \geq 0} (-1)^i \dim H^i_c(X, \mathbb{Q}_\ell). \end{aligned}$$

It has the following properties.

Properties 8.2.

(1) **Topological invariance:**(cf. [Mil13, Chapter 1, Section 2, p. 22]) For any $X \in \mathcal{V}_F$, we have $\chi(X) = \chi(X_{\text{red}})$, where X_{red} is the underlying reduced subscheme of X.
(2) **Additivity:**(cf. [Mil13, Chapter 1, Section 9, p. 65]) If $X = \bigsqcup Z_i$ is a decomposition of X into a disjoint union of locally closed subschemes, then $\chi(X) = \sum \chi(Z_i)$.
(3) **Dimension axiom:**(cf. [Mil13, Chapter 1, Section 9, p. 65]) $\chi(\mathrm{Spec}\ (F)) = 1$.
(4) **Cohomology of the projective line:**(cf. [Mil13, Chapter 1, Section 14, p. 90]) $\chi(\mathbb{P}^1_{\overline{F}}) = 2$. Combined with the dimension axiom and additivity, it follows that the compactly supported Euler characteristic of a tree of N projective lines over F is $N + 1$.
(5) **Étale Riemann-Hurwitz formula:**(cf. [Mil80, Chapter V, Remark 2.15 (c)]) Let Y be a regular F-scheme and let $f \colon X \to Y$ be a finite étale map of F-schemes of degree d. Then $\chi(X) = d\chi(Y)$.

Example 8.3. Let X be the curve with affine equation $y^n = x^n - \pi_K$. The geometric generic fiber $\mathcal{X}_{\overline{K}}$ is smooth of genus $g = \frac{(n-2)(n-1)}{2}$, and the special fiber $\mathcal{X}_k$ consists of n irreducible components isomorphic to $\mathbb{P}^1_k$ meeting pairwise transversely at a single point. By Properties 8.2, we see $\chi(\mathcal{X}_{\overline{K}}) = 3n - n^2$ and $\chi(\mathcal{X}_k) = n + 1$, and by Proposition 4.1, the Swan conductor $\delta(X/K)$ of X is 0, so that $\mathrm{Art}^+(X/K) = (n-1)^2$.

8.1. **Artin conductor of the normalization of a regular model of $\mathbb{P}^1_K$ with normal crossings branch divisor.** For the rest of §9, let X be the smooth projective K-curve with affine equation $y^n = f(x)$ satisfying Assumption 6.1. In §8.1, we make the following assumption:

Assumption 8.4. We are given a regular model $\mathcal{Y}$ of $\mathbb{P}^1_K$ whose normalization $\mathcal{X} \to \mathcal{Y}$ in $K(X) = K(x)[y]/(y^n - f(x))$ has branch locus B that is a simple normal crossings divisor.

Notation 8.5. Under Assumption 8.4, let R be the inverse image of B in $\mathcal{X}$.

- Write $\mathcal{Y}_s$ and $\mathcal{Y}_{\overline{\eta}} \cong \mathbb{P}^1_{\overline{K}}$ for the special and geometric generic fibers of $\mathcal{Y}$, respectively. Similarly define $B_s, B_{\overline{\eta}}, R_s$ and $R_{\overline{\eta}}$.
- For every $e \mid n$, let N_e be the number of irreducible components of $\mathcal{Y}_s$ with ramification index e. Let $N = \sum_{e|n} N_e$.
- For every pair of integers e, e' dividing n, let $c_{ee'}$ be the number of pairwise intersections of an irreducible component of B with ramification index e with another such component of ramification index e'.

Fix f as in Assumption 6.1. For each $1 \leq i \leq r$, recall that we have also chosen a root α_i of f_i and we set $K_i = K(\alpha_i)$.

Proposition 8.6. *Work under Assumption 8.4 and Notation 8.5. Then* $\mathrm{Art}^+(\mathcal{X}/\mathcal{O}_K)$ *equals*

$$n\,(N-1) - \sum_{e|n} \left(n - \frac{n}{e}\right) \left[2N_e - \sum_{\substack{i \\ \gcd(d_i,n)=n/e}} \Delta_{K_i/K}\right] + \sum_{\substack{e \leq e' \\ e|n, e'|n}} c_{ee'} \left[n + \gcd\left(\frac{n}{e}, \frac{n}{e'}\right) - \frac{n}{e} - \frac{n}{e'}\right].$$

Proof. From Definition 8.1 and Proposition 4.1, it suffices to show that $\chi(\mathcal{X}_s) - \chi(\mathcal{X}_{\overline{K}})$ equals

$$\begin{aligned} n\,(N-1) - \sum_{e|n} \left(n - \frac{n}{e}\right) &\left[2N_e - \sum_{\substack{i \\ \gcd(d_i,n)=\frac{n}{e}}} (\deg(f_i) - 1)\right] \\ &+ \sum_{\substack{e \leq e' \\ e|n, e'|n}} c_{ee'} \left[n + \gcd\left(\frac{n}{e}, \frac{n}{e'}\right) - \frac{n}{e} - \frac{n}{e'}\right] \end{aligned} \tag{8.7}$$

To prove this, we will use Properties 8.2 of χ with $F = \overline{K}$ and $F = k$. (We will omit specifying which of the two cases we are in since it is clear from context.)

Since $\mathrm{char}(k) \nmid n$, it follows that $\mathcal{X}_s \setminus R_s \to \mathcal{Y}_s \setminus B_s$ and $\mathcal{X}_{\overline{\eta}} \setminus R_{\overline{\eta}} \to \mathcal{Y}_{\overline{\eta}} \setminus B_{\overline{\eta}}$ are finite étale degree n maps. For $*$ in $\{s, \overline{\eta}\}$, using Property 8.2 (5) we get

$$\chi(\mathcal{X}_* \setminus R_*) = n\chi(\mathcal{Y}_* \setminus B_*),$$

and combining this with Property 8.2 (2) applied to the decompositions $\mathcal{X}_* = R_* \sqcup (\mathcal{X}_* \setminus R_*)$ and $\mathcal{Y}_* = B_* \sqcup (\mathcal{Y}_* \setminus B_*)$, we get

$$\begin{aligned} \chi(\mathcal{X}_s) - \chi(\mathcal{X}_{\overline{K}}) &= \chi(\mathcal{X}_s \setminus R_s) + \chi(R_s) - \chi(\mathcal{X}_{\overline{\eta}} \setminus R_{\overline{\eta}}) - \chi(R_{\overline{\eta}}) \\ &= n(\chi(\mathcal{Y}_s \setminus B_s) - \chi(\mathcal{Y}_{\overline{\eta}} \setminus B_{\overline{\eta}})) + \chi(R_s) - \chi(R_{\overline{\eta}}) \\ &= n(\chi(\mathcal{Y}_s) - \chi(B_s) - \chi(\mathcal{Y}_{\overline{\eta}}) + \chi(B_{\overline{\eta}})) + \chi(R_s) - \chi(R_{\overline{\eta}}) \\ &= n(\chi(\mathcal{Y}_s) - \chi(\mathcal{Y}_{\overline{\eta}})) + [n\chi(B_{\overline{\eta}}) - \chi(R_{\overline{\eta}})] + [-n\chi(B_s) + \chi(R_s)]\,. \end{aligned} \tag{8.8}$$

Since $\mathcal{Y}_{\overline{\eta}} \cong \mathbb{P}^1_{\overline{K}}$ and $\mathcal{Y}_s$ is a tree of N (possibly non-reduced) projective lines over k, using Properties 8.2 (1) and (4), we get

$$\chi(\mathcal{Y}_s) - \chi(\mathcal{Y}_{\overline{\eta}}) = (N+1) - 2 = N - 1. \tag{8.9}$$

Let $B^\circ \subset B$ be the open subset of the branch locus such that its complement is all pairwise intersections of components of B, and let R° be the preimage of B° under $\mathcal{X} \to \mathcal{Y}$. Then we have $B^\circ = \sqcup_{e|n} B^\circ_e$, where $B^\circ_e \subset B^\circ$ is the union of irreducible components of ramification index e. Let R°_e be the preimage of B°_e under $\mathcal{X} \to \mathcal{Y}$. Since, for all $e \mid n$, $R^\circ_e \to B^\circ_e$ is a topologically trivial map of degree e followed by an étale map of degree n/e, Properties 8.2 (1) and (5) show that

$$\chi((R_e^\circ)_s) = \frac{n}{e}\chi((B_e^\circ)_s), \tag{8.10}$$

and

$$\chi((R_e^\circ)_{\overline{\eta}}) = \frac{n}{e}\chi((B_e^\circ)_{\overline{\eta}}). \tag{8.11}$$

Since $n \mid \deg(f)$, it follows that $B_{\overline{\eta}}$ is precisely the zeroes of f and using Property 8.2 (3) and Assumption 6.1 we have

$$\chi((B_e)_{\overline{\eta}}) = \sum_{i,\gcd(d_i,n)=n/e} \deg(f_i), \tag{8.12}$$

and,

$$\chi(B_{\overline{\eta}}) = \sum_{e\mid n}\sum_{i,\gcd(d_i,n)=n/e} \deg(f_i). \tag{8.13}$$

Furthermore, since $B^\circ_{\overline{\eta}} = B_{\overline{\eta}}$, it also follows that $R^\circ_{\overline{\eta}} = R_{\overline{\eta}}$, and combining Equations (8.11) and (8.12) and adding over all $e \mid n$, we also get

$$-\chi(R_{\overline{\eta}}) = \sum_{e\mid n}\frac{n}{e}\left(-\sum_{i,\gcd(d_i,n)=n/e} \deg(f_i)\right). \tag{8.14}$$

Now $(B_e)_s$ is the union of N_e vertical components of ramification index e, each isomorphic to $\mathbb{P}^1_k$, and closed points, each isomorphic to Spec k, one for each of the specializations of the f_i that satisfy $\gcd(d_i, n) = n/e$. Using Property 8.2 (3), we see that the sum over all components of $(B_e)_s$ of the Euler characteristic of the intersection of each component of $(B_e)_s$ with components of B_s is $2c_{ee} + \sum_{e'\neq e} c_{ee'}$. Combining this with Property 8.2 (2) applied to this decomposition and Properties 8.2 (3) and 8.2 (4), we get

$$\chi((B_e^\circ)_s) = 2N_e - 2c_{ee} - \sum_{e'\neq e} c_{ee'} + \sum_{i,\gcd(d_i,n)=n/e} 1. \tag{8.15}$$

Combining Equations (8.10) and (8.15) and adding over all $e \mid n$, we also get

$$-n\chi((B^\circ)_s) + \chi((R^\circ)_s) = \sum_{e\mid n}\left(-n+\frac{n}{e}\right)\left[2N_e - 2c_{ee} - \sum_{e'\neq e} c_{ee'} + \sum_{i,\gcd(d_i,n)=n/e} 1\right]. \tag{8.16}$$

Now for every closed point of B that lies on the intersection of a component of ramification indices e and e', by Proposition 7.6(iii) we know that there are $\gcd(n/e, n/e')$ closed points above it in R. This implies that

$$-n\chi(B_s \setminus B_s^\circ) = \sum_{\substack{e\leq e'\\ e\mid n, e'\mid n}} -nc_{ee'}, \text{ and, } \chi(R_s \setminus R_s^\circ) = \sum_{\substack{e\leq e'\\ e\mid n, e'\mid n}} c_{ee'}\gcd\left(\frac{n}{e},\frac{n}{e'}\right),$$

and therefore

$$-n\chi(B_s \setminus B_s^\circ) + \chi(R_s \setminus R_s^\circ) = \sum_{\substack{e\leq e'\\ e\mid n, e'\mid n}} c_{ee'}\left(-n+\gcd\left(\frac{n}{e},\frac{n}{e'}\right)\right). \tag{8.17}$$

Adding n times Equations (8.9) and (8.13) to Equations (8.14), (8.16) and (8.17), applying Property 8.2 (2) to the decompositions $B_s = B_s^\circ \sqcup (B_s \setminus B_s^\circ), R_s = R_s^\circ \sqcup (R_s \setminus R_s^\circ)$ and rearranging the right hand side using the equality

$$\sum_{e|n}\left(n-\frac{n}{e}\right)\left(2c_{ee}+\sum_{e'\neq e}c_{ee'}\right)+\sum_{e\leq e'}c_{ee'}\left(-n+\gcd\left(\frac{n}{e},\frac{n}{e'}\right)\right)$$

$$=\sum_{\substack{e\leq e'\\ e|n,e'|n}}c_{ee'}\left[n+\gcd\left(\frac{n}{e},\frac{n}{e'}\right)-\frac{n}{e}-\frac{n}{e'}\right],$$

we get that the right hand side of Equation (8.8) equals the expression in (8.7). □

Let $\mathcal{Y}$ be as in Assumption 8.4. Let $\mathcal{X}\to\mathcal{Y}$ be the normalization of $\mathcal{Y}$ in $K(X)$. Let $z_1, ..., z_l$ be the points of multiplicity 2 in the branch divisor B of $\mathcal{X}\to\mathcal{Y}$. For $1\leq i\leq l$, let e_i, e_i' denote the ramification indices of the two irreducible components of B meeting at z_i.

Proposition 8.18. *Let $\mathcal{X}'\to\mathcal{X}$ be the minimal regular resolution of $\mathcal{X}$. Then*

$$\mathrm{Art}^+(\mathcal{X}'/\mathcal{O}_K)-\mathrm{Art}^+(\mathcal{X}/\mathcal{O}_K)=\sum_{i=1}^{l}\gcd\left(\frac{n}{e_i},\frac{n}{e_i'}\right)\cdot\mathrm{HJL}(z_i).$$

Proof. Let $T\subset\mathcal{X}_s$ be the set of singular points of $\mathcal{X}$ and let $T'\subset\mathcal{X}_s'$ be the preimage of the singular points under $\mathcal{X}'\to\mathcal{X}$. Since the map $\mathcal{X}'\to\mathcal{X}$ induces isomorphisms $\mathcal{X}'_{\overline{\eta}}\to\mathcal{X}_{\overline{\eta}}$ and $\mathcal{X}_s'\setminus T'\to\mathcal{X}_s\setminus T$, from Property 8.2 (2) applied to the decompositions $\mathcal{X}_s=T\sqcup(\mathcal{X}_s\setminus T)$ and $\mathcal{X}_s'=T'\sqcup(\mathcal{X}_s'\setminus T')$, it suffices to show that

$$\chi(T')-\chi(T)=\sum_{i=1}^{l}\gcd\left(\frac{n}{e_i},\frac{n}{e_i'}\right)\cdot\mathrm{HJL}(z_i)$$

We now describe T and T'. By Proposition 7.6(i,iii), the singularities of $\mathcal{X}$ are isolated, and lie above the points z_i. Furthermore by Remark 7.10, if x_i is a point above z_i, then the reduced exceptional divisor of the minimal resolution of x_i is a chain of $\mathrm{HJL}(z_i)$ components isomorphic to $\mathbb{P}^1_k$, and by Proposition 7.6(iii), there are exactly $\gcd\left(\frac{n}{e_i},\frac{n}{e_i'}\right)$ such points x_i for each z_i. The proposition now follows from Properties 8.2 (2) and (4). □

9. Rephrasing the conductor-discriminant inequality

The goal of this section is to prove Proposition 9.4, where we use Proposition 8.6 and Proposition 8.18 to rephrase the conductor-discriminant inequality (1.5) in terms of a quantity called the *conductor discriminant contribution* (cdc) attached to the resolution of each isolated singularity on a normal model of X (see Definition 9.3 and Proposition 9.4). Similarly, in Corollary 9.14, we rephrase the conductor exponent-discriminant inequality in terms of a quantity called the *conductor exponent discriminant contribution* (cedc).

9.1. **Conductor-discriminant contribution of a sequence of blowups.** For the rest of §9, we drop Assumption 8.4 but still assume X is given by an equation $y^n = f(x)$ as in Assumption 6.1.

Notation 9.1. (Branch divisor) Let $\mathcal{Y}$ be a regular model of $\mathbb{P}^1_K$. Let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $K(X)$. If Γ is an irreducible divisor on $\mathcal{Y}$, the ramification index of Γ always means the ramification index with respect to $\mathcal{X} \to \mathcal{Y}$; likewise, the branch divisor on $\mathcal{Y}$ always means the branch divisor of $\mathcal{X} \to \mathcal{Y}$.

Finally, for a divisor $E = \sum a_i \Gamma_i$ on $\mathcal{Y}$, let $(E \bmod n)^{\mathrm{red}}$ denote the divisor $\sum b_i \Gamma_i$ on $\mathcal{Y}$, where $b_i = 0$ if $n \mid a_i$ and $b_i = 1$ otherwise. For example, if $E = \mathrm{div}_0(f)$ on $\mathcal{Y}$, then $(E \bmod n)^{\mathrm{red}}$ is the branch divisor of $\mathcal{X} \to \mathcal{Y}$.

Notation 9.2. (Multiplicities and ramification indices) Let $\mathcal{Y}$ be a regular model of $\mathbb{P}^1_K$, and let F be an effective divisor on $\mathcal{Y}$ that is supported on components of $\mathrm{div}_0(f)$. Let $\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0 = \mathcal{Y}$ be a finite sequence of blowups at reduced closed points. For $0 \le k \le m$, let F_k be the inverse image of F under $\mathcal{Y}_k \to \mathcal{Y}_0$, and let $D_k = (F_k \bmod n)^{\mathrm{red}}$. For $1 \le k \le m$, let $y_k \in \mathcal{Y}_{k-1}$ denote the center of the blowup $\mathcal{Y}_k \to \mathcal{Y}_{k-1}$, and let $\nu_k := \mu_{y_k}(F_{k-1})$ and $\mu_k := \mu_{y_k}(D_{k-1})$. Let $z_1, \ldots, z_l$ be the points of multiplicity 2 in D_m at which D_m has a simple normal crossing, and for $1 \le i \le l$, let e_i, e_i' be the ramification indices of the two irreducible components of D_m meeting at z_i.

Definition 9.3. Let $\mathcal{Y}_i, z_i, \nu_k, \mu_k, e_i$ be as in Notation 9.2. The *conductor-discriminant contribution* associated to the sequence of blowups $\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0$ and divisor F is defined by

$$\begin{aligned}\mathrm{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F) := &\sum_{k : n \nmid \nu_k} (n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3)) \\ &+ \sum_{k : n \mid \nu_k} (-n + (n-1)\mu_k(\mu_k - 1)) \\ &+ \sum_{i=1}^{l} \left(\left(\frac{n}{e_i} - 1 \right) + \left(\frac{n}{e_i'} - 1 \right) + \left(n - \gcd\left(\frac{n}{e_i}, \frac{n}{e_i'} \right) (\mathrm{HJL}(z_i) + 1) \right) \right).\end{aligned}$$

Similarly, the *conductor exponent-discriminant contribution* is defined by

$$\begin{aligned}\mathrm{cedc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F) := &\sum_{k : n \nmid \nu_k} (1 + n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3)) \\ &+ \sum_{k : n \mid \nu_k} (1 - n + (n-1)\mu_k(\mu_k - 1)) \\ &+ \sum_{i=1}^{l} \left(\left(\frac{n}{e_i} - 1 \right) + \left(\frac{n}{e_i'} - 1 \right) + \left(n - \gcd\left(\frac{n}{e_i}, \frac{n}{e_i'} \right) \right) \right).\end{aligned}$$

This terminology is motivated by the following proposition.

Proposition 9.4. *Let $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}_K}$, and let $\mathcal{Y}_0, \mathcal{Y}_m, \mathcal{X}_m$ be as in Notation 9.2. Let $F := \mathrm{div}_0(f) \subset \mathcal{Y}_0$. Assume that the branch divisor of the normalization $\mathcal{X}_m \to \mathcal{Y}_m$ of $\mathcal{Y}_m$ in*

$K(X)$ *has simple normal crossings. If* $\mathcal{X}'_m \to \mathcal{X}_m$ *is the minimal regular resolution, then*

$$\begin{aligned}(n-1)\nu_K(\mathrm{disc}_n(f)) - \mathrm{Art}^+(\mathcal{X}'_m) &= \mathrm{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F) \\ &\quad + \sum_{i=1}^{r}(\gcd(n,d_i)-1)\Delta_{K_i/K} + \begin{cases} 0 & b=0 \\ 2-2\gcd(b,n) & b \geq 1 \end{cases} \\ &\geq \mathrm{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F) + \begin{cases} 0 & b=0 \\ 2-2\gcd(b,n) & b \geq 1. \end{cases}\end{aligned}$$

We first prove a few lemmas. Let $\mathcal{Y}$ be a regular model of $\mathbb{P}^1_K$. Let D be the branch divisor of the normalization $\mathcal{X} \to \mathcal{Y}$ of $\mathcal{Y}$ in $K(X)$. Let $y \in \mathcal{Y}$ be a closed point such that $\mu_y D = \mu$ and $\mu_y(\mathrm{div}_0(f)) = \nu$. Let $\pi\colon \mathcal{Y}' \to \mathcal{Y}$ be the blowup of $\mathcal{Y}$ at y, let E be the corresponding exceptional divisor with normalized discrete valuation $\mathrm{ord}_E\colon K(Y)^\times \to \mathbb{Z}$, and let D' be the branch divisor of the normalization $\mathcal{X}' \to \mathcal{Y}'$ of $\mathcal{Y}'$ in $K(X)$. Let $\tilde{D}$ and $\tilde{D}'$ denote the normalizations of the divisors D and D' respectively. We continue to use D to denote the strict transform in $\mathcal{Y}'$ of the divisor $D \subset \mathcal{Y}$.

Lemma 9.5.

(i) $\mathrm{ord}_E(f) = \nu$. *If* $n \mid \nu$, *then* $D' = D$, *and if* $n \nmid \nu$, *then* $D' = D + E$.

(ii)
$$\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}}/\mathcal{O}_D) = \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}'}/\mathcal{O}_{D'}) + \begin{cases} \frac{\mu(\mu-1)}{2} & n \mid \nu \\ \frac{\mu(\mu-3)}{2} & n \nmid \nu. \end{cases}$$

Proof. By definition of X, D, and D', it follows that $D \equiv (\mathrm{div}_0(f) \bmod n)^{\mathrm{red}} \subset \mathcal{Y}$ and $D' \equiv (\mathrm{div}_0(f) \bmod n)^{\mathrm{red}} \subset \mathcal{Y}'$. By [Liu06, Proposition 9.2.23], it follows that $\mathrm{ord}_E(f) = \mu_y(\mathrm{div}_0(f)) =: \nu$, proving (i). From the last two sentences, it follows that if $n \mid \nu$, then D' is the strict transform of D, and if $n \nmid \nu$, then $D' = D + E$, where we continue to use D to denote the strict transform in $\mathcal{Y}'$ of the divisor $D \subset \mathcal{Y}$. The lemma then follows from [OS24, Equations (3.6) and (3.7)], where we ignore the final inequalities in both equations, which are the only parts of these equations that depend on the assumption $n = 2$ in [OS24]. □

Corollary 9.6. *Let* D_0, D_m *be as in Notation 9.2. Assume that* D_m *has simple normal crossings and that* l *is the number of multiplicity* 2 *points of* D_m. *Then*

$$\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_0}/\mathcal{O}_{D_0}) = l + \sum_{k: n\nmid\nu_k} \frac{\mu_k(\mu_k-3)}{2} + \sum_{k: n\mid\nu_k} \frac{\mu_k(\mu_k-1)}{2}.$$

Proof. Since D_m has simple normal crossings and l is the number of multiplicity 2 points of D_m, it follows that $\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_m}/\mathcal{O}_{D_m}) = l$. Combining this with a repeated application of Lemma 9.5, we get

$$\begin{aligned}\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_0}/\mathcal{O}_{D_0}) &= \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_m}/\mathcal{O}_{D_m}) + \sum_{k: n\nmid\nu_k} \frac{\mu_k(\mu_k-3)}{2} + \sum_{k: n\mid\nu_k} \frac{\mu_k(\mu_k-1)}{2} \\ &= l + \sum_{k: n\nmid\nu_k} \frac{\mu_k(\mu_k-3)}{2} + \sum_{k: n\mid\nu_k} \frac{\mu_k(\mu_k-1)}{2}.\end{aligned}$$

□

Lemma 9.7. *Suppose $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}_K}$, and let D_0 be as in Notation 9.2. Then*

$$\mathrm{db}_K(f) = 2\,\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_0}/\mathcal{O}_{D_0}) - \begin{cases} 0 & b = 0 \\ 2 & b \geq 1 \end{cases}$$

Proof. For convenience, set $c = 0$ if $b = 0$ and $c = 1$ otherwise. Let D_0' be the horizontal part of D_0, and let $\widetilde{D_0'}$ denote the normalization of D_0'. In (9.8) below, the first equality follows from Remark 5.4, the second from the equation between (3.9) and (3.10) in [OS24], and the third from [OS24, (3.9)].

$$\tag{9.8} \begin{aligned} \mathrm{db}_K(f) &= 2c(\deg(\mathrm{rad}(f)) - 1) + \mathrm{db}_K(f/\pi^b) \\ &= 2c(\deg(\mathrm{rad}(f)) - 1) + 2\ \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\widetilde{D_0'}}/\mathcal{O}_{D_0'}) \\ &= 2c(\deg(\mathrm{rad}(f)) - 1) + 2\ \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_0}/\mathcal{O}_{D_0}) - 2c\deg(\mathrm{rad}(f)) \\ &= 2\ \mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_{\tilde{D}_0}/\mathcal{O}_{D_0}) - 2c. \end{aligned}$$

□

Proof of Proposition 9.4. Keep the notation of Definition 9.3. For $e \mid n$, let N_e be the number of irreducible components of the special fiber $(\mathcal{Y}_m)_s$ of ramification index e, and let $N := \sum_{e|n} N_e$. By Propositions 8.6, 8.18 and 7.6(iii), we have

$$\tag{9.9} \begin{aligned} \mathrm{Art}^+(\mathcal{X}_m') = n(N-1) &- \sum_{e|n}\left(n - \frac{n}{e}\right)\left(2N_e - \sum_{i:\gcd(d_i,n)=\frac{n}{e}} \Delta_{K_i/K}\right) \\ &+ \sum_{i=1}^{l}\left[n + \gcd\left(\frac{n}{e_i}, \frac{n}{e_i'}\right) - \frac{n}{e_i} - \frac{n}{e_i'}\right] + \sum_{i=1}^{l} \gcd\left(\frac{n}{e_i}, \frac{n}{e_i'}\right)\mathrm{HJL}(z_i). \end{aligned}$$

The special fiber $(\mathcal{Y}_0)_s = \mathbb{P}^1_k$ is irreducible and has ramification index $\frac{n}{\gcd(n,b)}$, where b is the valuation of the leading coefficient of f as in Assumption 6.1. By Lemma 9.5 (i), the blowup $\mathcal{Y}_k \to \mathcal{Y}_{k-1}$, centered at a closed point in $\mathcal{Y}_{k-1}$ of multiplicity ν_k in $\mathrm{div}_0(f)$, introduces a single irreducible component in the special fiber on which the order of f is ν_k, which implies that the ramification index of this component is $\frac{n}{\gcd(n,\nu_k)}$. It follows that $N = m + 1$,

$$\sum_{e|n} 2N_e\left(n - \frac{n}{e}\right) = 2(n - \gcd(n,b)) + \sum_{k=1}^{m} 2(n - \gcd(\nu_k, n)),$$

and therefore

$$\tag{9.10} n(N-1) - \sum_{e|n} 2N_e\left(n - \frac{n}{e}\right) = -2(n - \gcd(n,b)) + \sum_{k=1}^{m}(2\gcd(\nu_k, n) - n).$$

We also have

$$\sum_{e|n}\left(n-\frac{n}{e}\right)\left(\sum_{i:\gcd(d_i,n)=\frac{n}{e}}\Delta_{K_i/K}\right)=\sum_{i=1}^{r}(n-\gcd(n,d_i))\Delta_{K_i/K} \tag{9.11}$$
$$=(n-1)\sum_{i=1}^{r}\Delta_{K_i/K}-\sum_{i=1}^{r}(\gcd(n,d_i)-1)\Delta_{K_i/K}.$$

Putting (9.9), (9.10), and (9.11) together, we get

$$\begin{aligned}\mathrm{Art}^{+}(\mathcal{X}'_m)&=-2(n-\gcd(n,b))+\sum_{k=1}^{m}(2\gcd(\nu_k,n)-n)\\&+(n-1)\sum_{i=1}^{r}\Delta_{K_i/K}-\sum_{i=1}^{r}(\gcd(d_i,n)-1)\Delta_{K_i/K}\\&+\sum_{i=1}^{l}\left(n-\frac{n}{e_i}-\frac{n}{e'_i}+\gcd\left(\frac{n}{e_i},\frac{n}{e'_i}\right)(\mathrm{HJL}(z_i)+1)\right).\end{aligned} \tag{9.12}$$

Using (9.12) and Definition 5.1 for the first equality, Corollary 9.6 and Lemma 9.7 for the second equality, and Definition 9.3 for the final equality, we get that

$$\begin{aligned}&(n-1)\nu_K(\mathrm{disc}_n(f))-\mathrm{Art}^{+}(\mathcal{X}'_m)\\&=(n-1)\,\mathrm{db}_K(f)+2(n-\gcd(n,b))-\sum_{k=1}^{m}(2\gcd(n,\nu_k)-n)+\sum_{i=1}^{r}(\gcd(n,d_i)-1)\Delta_{K_i/K}\\&\quad-\sum_{i=1}^{l}\left(n-\frac{n}{e_i}-\frac{n}{e'_i}+\gcd\left(\frac{n}{e_i},\frac{n}{e'_i}\right)(\mathrm{HJL}(z_i)+1)\right)\\&=2l(n-1)+\sum_{k:n\nmid\nu_k}(n-1)\mu_k(\mu_k-3)+\sum_{k:n|\nu_k}(n-1)\mu_k(\mu_k-1)-\begin{cases}0 & b=0\\2(n-1) & b\geq 1\end{cases}\\&\quad+2(n-\gcd(n,b))-\sum_{k=1}^{m}(2\gcd(n,\nu_k)-n)+\sum_{i=1}^{r}(\gcd(n,d_i)-1)\Delta_{K_i/K}\\&\quad-\sum_{i=1}^{l}\left(n-\frac{n}{e_i}-\frac{n}{e'_i}+\gcd\left(\frac{n}{e_i},\frac{n}{e'_i}\right)(\mathrm{HJL}(z_i)+1)\right)\\&=\sum_{k:n\nmid\nu_k}(n-2\gcd(\nu_k,n)+(n-1)\mu_k(\mu_k-3))+\sum_{k:n|\nu_k}(-n+(n-1)\mu_k(\mu_k-1))\\&\quad+\sum_{i=1}^{l}\left(\frac{n}{e_i}-1\right)+\left(\frac{n}{e'_i}-1\right)+\left(n-\gcd\left(\frac{n}{e_i},\frac{n}{e'_i}\right)(\mathrm{HJL}(z_i)+1)\right)\\&\quad+\sum_{i=1}^{r}(\gcd(n,d_i)-1)\Delta_{K_i/K}+2(n-\gcd(b,n))-\begin{cases}0 & b=0\\2(n-1) & b\geq 1\end{cases}\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{cdc}(\mathcal{Y}_m \to ...\mathcal{Y}_0, F) + \sum_{i=1}^{r} (\gcd(n, d_i) - 1)\Delta_{K_i/K} + \begin{cases} 0 & b = 0 \\ 2 - 2\gcd(b, n) & b \geq 1 \end{cases} \\
&\geq \operatorname{cdc}(\mathcal{Y}_m \to ...\mathcal{Y}_0, F) + \begin{cases} 0 & b = 0 \\ 2 - 2\gcd(b, n) & b \geq 1, \end{cases}
\end{aligned}
$$

where we use that the terms $(\gcd(n, d_i) - 1)\Delta_{K_i/K}$ are nonnegative for the last inequality. □

Now let $\varphi(X/K)$ denote the conductor exponent for the Galois representation $\operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}_{\mathbb{Q}_\ell}(H^1_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_\ell))$ ($\ell \neq \operatorname{char} k$), which depends only on the generic fiber X and not on the model $\mathcal{X}$. We can adapt Proposition 9.4 to give a lower bound on $(n-1)\nu_K(\operatorname{disc}_n(f)) - \varphi(X/K)$. Recall from Proposition 1.8 that if $\mathcal{X}/\mathcal{O}_K$ is any regular model of X and N is the number of irreducible components of the special fiber $\mathcal{X}_s$, then

$$
\operatorname{Art}^+(\mathcal{X}/\mathcal{O}_K) = N - 1 + \varphi(X/K). \tag{9.13}
$$

Corollary 9.14. *Keep the notation of Proposition 9.4. Then*

$$
(n-1)\nu_K(\operatorname{disc}_n(f)) - \varphi(X/K) \geq \operatorname{cedc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to ... \to \mathcal{Y}_0, F) + \begin{cases} 0 & b = 0 \\ 2 - 2\gcd(b, n) & b \geq 1. \end{cases}
$$

Proof. Let N'_m denote the number of irreducible components of the special fiber $(\mathcal{X}'_m)_s$. By Proposition 9.4 and (9.13) applied to the regular model $\mathcal{X}'_m$ of X, it follows that

$$
\begin{aligned}
&(n-1)\nu_K(\operatorname{disc}_n(f)) - \varphi(X/K) \\
&= N'_m - 1 + (n-1)\nu_K(\operatorname{disc}_n(f)) - \operatorname{Art}^+(\mathcal{X}'_m) \\
&\geq N'_m - 1 + \operatorname{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to ... \to \mathcal{Y}_0, F) + \begin{cases} 0 & b = 0 \\ 2 - 2\gcd(b, n) & b \geq 1. \end{cases}
\end{aligned} \tag{9.15}
$$

We now estimate N'_m. The strict transforms on $\mathcal{Y}_m$ of the special fiber $(\mathcal{Y}_0)_s = \mathbb{P}^1_k$ and of the exceptional divisor of each blowup $\mathcal{Y}_k \to \mathcal{Y}_{k-1}$ are distinct irreducible components of $(\mathcal{Y}_m)_s$. In turn, each irreducible component of $(\mathcal{Y}_m)_s$ lies under at least one irreducible component of the special fiber $(\mathcal{X}_m)_s$ of the normalization $\mathcal{X}_m$ of $\mathcal{Y}_m$ in $K(X)$. Now let $z \in \mathcal{Y}_m$ be the intersection point of two irreducible components Γ, Γ' of the branch divisor D_m with ramification indices e, e' respectively. By Proposition 7.6(iii), Definition 7.9, and Remark 7.10, there are $\gcd(\frac{n}{e}, \frac{n}{e'})$ points of $\mathcal{X}_m$ lying over z, and for each such point x, there are $\operatorname{HJL}(z_i)$ irreducible components of the special fiber $(\mathcal{X}'_m)_s$ contained in the preimage of x under the resolution $\mathcal{X}'_m \to \mathcal{X}_m$. It follows that

$$
N'_m \geq 1 + m + \sum_{i=1}^{l} \gcd\left(\frac{n}{e}, \frac{n}{e'}\right) \cdot \operatorname{HJL}(z_i).
$$

Combining this with the definition of $\mathrm{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F)$ and writing $m = \sum_{k:\, n\nmid\nu_k} 1 + \sum_{k:\, n|\nu_k} 1$, we get

$$\begin{aligned}
&N'_m - 1 + \mathrm{cdc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, F)\\
=&N'_m - 1 + \sum_{k:n\nmid\nu_k} [n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3)] + \sum_{k:n|\nu_k} [-n + (n-1)\mu_k(\mu_k - 1)]\\
&\quad + \sum_{i=1}^{l} \left[\left(\frac{n}{e_i} - 1\right) + \left(\frac{n}{e'_i} - 1\right) + \left(n - \gcd\left(\frac{n}{e_i}, \frac{n}{e'_i}\right)(\mathrm{HJL}(z_i) + 1)\right) \right]\\
\geq & \sum_{k:n\nmid\nu_k} [1 + n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3)]\\
&\quad + \sum_{k:n|\nu_k} [1 - n + (n-1)\mu_k(\mu_k - 1)] + \sum_{i=1}^{l} \left[\left(\frac{n}{e_i} - 1\right) + \left(\frac{n}{e'_i} - 1\right) + \left(n - \gcd\left(\frac{n}{e_i}, \frac{n}{e'_i}\right)\right) \right]\\
=&\, \mathrm{cedc}(\mathcal{Y}_m \to \mathcal{Y}_{m-1} \to \ldots \to \mathcal{Y}_0, \mathcal{F}).
\end{aligned}$$

Substituting this into Equation (9.15), we obtain the corollary. ∎

10. Proof of the conductor-discriminant inequality via explicit analysis of multiplicity 2 points of the branch divisor

In this section, we first prove the conductor exponent-discriminant inequality (Corollary 1.9), which is easier to prove, and then prove the full conductor-discriminant inequality (Theorem 1.4). Recall from Lemma 2.1 that when the branch divisor of $\mathcal{X}_0 \to \mathcal{Y}_0$ is smooth, or equivalently, when $\nu_K(\mathrm{disc}_n(f)) = 0$, the normalization $\mathcal{X}_0$ of $\mathcal{Y}_0 := \mathbb{P}^1_{\mathcal{O}_K}$ in $K(X)$ is smooth, and hence the Artin conductor of $\mathcal{X}_0$ is 0. When $\nu_K(\mathrm{disc}_n(f)) \neq 0$, the problem reduces to explicitly constructing a model $\mathcal{X}_m$ as the normalization of an inductively defined blowup $\mathcal{Y}_m$ of $\mathcal{Y}_0$ that partially resolves the branch locus, and such that $\mathcal{X}_m$ has only tame cyclic quotient singularities (Proposition 10.8), and which has an explicit resolution $\mathcal{X}'_m$ with sufficiently many contractible components (Proposition 10.15). Proposition 10.8 reduces the proof of Corollary 1.9 and Theorem 1.4 to the problem of explicitly understanding resolutions of multiplicity 2 points of the branch divisor. We define explicit resolutions of these multiplicity 2 points (Definition 10.2 and Lemma 10.3), and then appeal to key computations of the conductor-discriminant contribution and the conductor exponent-discriminant contribution for multiplicity 2 points from [Ste26, Theorems 1.4,1.6] to deduce our results.

Notation 10.1. Let $\mathcal{Y}, \mathcal{X}$ be as in Notation 9.1. Let $F := \mathrm{div}_0(f) \subset \mathcal{Y}$ and let $D := (F \bmod n)^{\mathrm{red}}$ be the branch divisor for the map $\mathcal{X} \to \mathcal{Y}$. Let y be a multiplicity 2 point of D, and let F_y be the sum of the irreducible components of D that contain y, with multiplicities inherited from F.

Definition 10.2. The *local n-resolution* of D at y is the sequence of blowups $\ldots \to \mathcal{Y}_k \to \ldots \to \mathcal{Y}_0 =: \mathcal{Y}$ centered at reduced closed points $y_{k+1} \in \mathcal{Y}_k$ defined inductively as follows.

- Set $F'_0 = F_0 = F_y$.
- If the divisor F'_k on $\mathcal{Y}_k$ contains multiple points lying over y via the map $\mathcal{Y}_k \to \mathcal{Y}_0$, or if F'_k contains a unique point lying over y and the divisor $(F_k \bmod n)^{\mathrm{red}}$ has simple

normal crossings at this point, the local n-resolution of D at y terminates at the stage $\mathcal{Y}_k \to \ldots \to \mathcal{Y}_0$.
- Otherwise, let $y_{k+1} \in \mathcal{Y}_k$ be the unique point of F'_k lying over y, let $\mathcal{Y}_{k+1} \to \mathcal{Y}_k$ be the blowup of $\mathcal{Y}_k$ at y_{k+1}, and let F_{k+1}, F'_{k+1} respectively denote the inverse image and strict transform of F_y under $\mathcal{Y}_{k+1} \to \mathcal{Y}_0$, so that in particular, F'_{k+1} contains at least one point lying over y.

Lemma 10.3. [Ste26, Proposition 2.14] *Keep the notation of Definition 10.2. The local n-resolution of D at y terminates at a finite stage $\mathcal{Y}_m \to \ldots \to \mathcal{Y}_0$ at which $(F_m \bmod n)^{\mathrm{red}}$ has simple normal crossings.*

The local n-resolution of a general multiplicity 2 point is described explicitly in [Ste26, Section 3]. We include a simple example to illustrate.

Example 10.4. Suppose $\mathrm{char}(k) \neq 3$, and let X be the smooth projective curve over K with affine equation $w^3 = x^3 - \pi_K^2$. Let $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}_K}$, let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $K(X)$ and let D be the branch divisor of $\mathcal{X} \to \mathcal{Y}$. Then D is the flat closure of the point of $\mathbb{P}^1_K$ defined by the ideal $(x^3 - \pi_K^2) \subset K[x]$, the unique closed point y on D is defined by the ideal $(x, \pi_K) \subset \mathcal{O}_K[x]$, and y is a singular point of D of multiplicity 2.

An explicit computation shows that the local 3-resolution terminates after two blow-ups $\mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{Y}$. Furthermore, if E_1, E_2 are the exceptional divisors of $\mathcal{Y}_1 \to \mathcal{Y}_0$ and $\mathcal{Y}_2 \to \mathcal{Y}_1$ respectively, and if E'_1 is the strict transform of E_1 on $\mathcal{Y}_2$, then

$$\begin{aligned} F_0 &= F'_0 = F_y = D, & (F_0 \bmod 3)^{\mathrm{red}} &= F'_0, \\ F_1 &= F'_1 + 2E_1, & (F_1 \bmod 3)^{\mathrm{red}} &= F'_1 + E_1, \\ F_2 &= F'_2 + 2E'_1 + 3E_2, & (F_2 \bmod 3)^{\mathrm{red}} &= F'_2 + E'_1. \end{aligned}$$

For more details of the computation above, apply [Ste26, Propositions 3.7, 3.12, and 3.14] with $n = e_0 = 3$ and $j = 1$. Note that in this case the local 3-resolution of y differs from the full embedded resolution of y since the total transform F_2 of D is not a simple normal crossings divisor; in this case the full embedded resolution of y needs one additional blowup (see, e.g., [Har77, Chapter V, Example 3.9.1] for the analogous computation in the geometric setting of a surface singularity over $\mathbb{C}$).

Definition 10.5. Keep the notation of Definition 10.2 and recall Definition 9.3. Define the *conductor-discriminant contribution* $\mathrm{cdc}(y)$ and *conductor exponent-discriminant contribution* $\mathrm{cedc}(y)$ of y to be those associated to the local n-resolution $\mathcal{Y}_m \to \ldots \to \mathcal{Y}_0$ of D at y and divisor F_y on $\mathcal{Y}$.

Proposition 10.6. [Ste26, Proposition 4.7] *Let D be as in Notation 10.1. Let y be a multiplicity 2 point of D contained in two regular irreducible components of D, whose ramification indices are e and e'. If y is a simple normal crossings point of D, or if $e \neq e'$, then* $\mathrm{cedc}(y) \geq \mathrm{cdc}(y) \geq 2(n/e) - 2$.

We now explain why resolutions of multiplicity 2 points are the bottlenecks for deducing the conductor-discriminant and conductor exponent-discriminant inequalities easily from the results thus far. We begin by showing that we can first resolve the points of multiplicity ≥ 3 in the branch divisor until there are no such points left, and we then construct explicit resolutions of points of multiplicity 2 one by one. To that end, we define the following arithmetic schemes. For the remainder of the paper, for an $\mathcal{O}_K$-model $\mathcal{Y}_i$ of $\mathbb{P}^1_K$, we let $\mathcal{X}_i$ denote the normalization of $\mathcal{Y}_i$ in $K(X)$ and we let D_i denote the branch divisor of the map $\mathcal{X}_i \to \mathcal{Y}_i$.

Lemma 10.7. *There is a sequence of blowups $\mathcal{Y}_j \to \mathcal{Y}_{j-1} \to \ldots \to \mathcal{Y}_0 := \mathcal{Y}$ at reduced closed points $y_k \in \mathcal{Y}_{k-1}$, such that*

(i) *for all $1 \leq k \leq j$, the point y_k has multiplicity ≥ 3 in the branch divisor D_{k-1}, and,*
(ii) *there are no points of multiplicity ≥ 3 in the branch divisor D_j.*

Proof. Since $\mathcal{O}_K$ is excellent, by the theorem on the existence of embedded resolutions of curves on arithmetic surfaces [Liu06, Theorems 9.2.26 and 9.2.2], there is a modification $\pi\colon \mathcal{Y}' \to \mathcal{Y}$, where π is given as a composition

$$\mathcal{Y}' =: \mathcal{Y}'_r \xrightarrow{\pi'_r} \mathcal{Y}'_{r-1} \xrightarrow{\pi'_{r-1}} \cdots \xrightarrow{\pi'_1} \mathcal{Y}'_0 := \mathcal{Y}$$

of blowups at reduced closed points $y'_i \in \mathcal{Y}'_{i-1}$, such that π^*D has normal crossings. In particular, the divisor $(\pi^*D)^{\mathrm{red}}$ contains no points of multiplicity ≥ 3. Since the branch divisor D' of the normalization $\mathcal{X}' \to \mathcal{Y}'$ of $\mathcal{Y}'$ in $K(X)$ satisfies $D' \leq (\pi^*D)^{\mathrm{red}}$, then D' also contains no points of multiplicity ≥ 3, and the sequence of blowups $\mathcal{Y}'_r \to \ldots \to \mathcal{Y}'_0$ satisfies (ii).

It is possible however that some y'_i has multiplicity < 3 in the appropriate branch divisor, and thus part (i) is not satisfied. So the idea is to do the process above, but without doing any of the blowups at points of multiplicity < 3 in the appropriate branch divisor. More precisely, we prove our general lemma by induction on r, the minimal number of point blowups required for the pullback of D to have normal crossings. If $r = 0$, then D has normal crossings and thus no points of multiplicity ≥ 3, so we can take $\mathcal{Y}_j = \mathcal{Y}$. In fact, regardless of r, if D has no points of multiplicity ≥ 3, we can take $\mathcal{Y}_j = \mathcal{Y}$.

Thus we may assume that D has a point y of multiplicity ≥ 3. Since y is an isolated singularity, we may further assume that $y = y'_0$, that is, $\pi'_1\colon \mathcal{Y}'_1 \to \mathcal{Y}$ is the blowup at y. Set $\mathcal{Y}_1 = \mathcal{Y}'_1$. Let $\pi' := \pi'_r \circ \cdots \circ \pi'_2$. Then the pullback $(\pi')^*D_1$ of the branch divisor D_1 has normal crossings since $(\pi')^*D_1 \leq (\pi')^*(\pi_1^*D)$ and since $(\pi')^*(\pi_1^*D) = (\pi' \circ \pi'_1)^*D$ has normal crossings by definition of the π'_i. and since $D_1 \leq (\pi'_1)^*D$. By induction, there is a sequence of point blowups of $\mathcal{Y}_1$ satisfying parts (i) and (ii) of the lemma, so we are done. ☐

Let $\mathcal{Y}_j$ be a model satisfying the assumptions in Lemma 10.7, and let $z_1, \ldots, z_w$ be the points of multiplicity 2 in $D_j \subset \mathcal{Y}_j$. For $1 \leq i \leq w$, we inductively define schemes $\mathcal{Y}_{j+i}$ such that there is a birational morphism $\mathcal{Y}_{j+i} \to \mathcal{Y}_j$ that is an isomorphism away from the set $\{z_1, \ldots, z_{i-1}\}$ in $\mathcal{Y}_j$ and its preimage in $\mathcal{Y}_{j+i}$. In particular, above each point in the set $\{z_i, \ldots, z_w\}$, there will be a unique point in $\mathcal{Y}_{j+i-1}$ also of multiplicity 2 in the corresponding branch divisor D_{j+i-1}. Define $\mathcal{Y}_{j+i} \to \mathcal{Y}_{j+i-1}$ to be the local n-resolution of the branch divisor D_{j+i-1} at the unique point of $\mathcal{Y}_{j+i-1}$ lying over z_i. Let $m = j + w$.

Proposition 10.8. *Suppose $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}_K}$. Let $\mathcal{Y}_j$ be as in Lemma 10.7, and define $\mathcal{Y}_m, \mathcal{X}_m, D_m$ as above. Then the branch divisor D_m of $\mathcal{X}_m \to \mathcal{Y}_m$ has simple normal crossings, the model $\mathcal{X}_m$ has only tame cyclic quotient singularities, and if $\mathcal{X}'_m \to \mathcal{X}_m$ is the minimal regular resolution, then there is a subset $A \subseteq \{z_1, \ldots, z_w\}$ such that*

$$(n-1)\nu_K(\mathrm{disc}_n(f)) - \mathrm{Art}^+(\mathcal{X}'_m) \geq \sum_{z \in A} \mathrm{cdc}(z),$$

and

$$(n-1)\nu_K(\mathrm{disc}_n(f)) - \varphi(X/K) \geq \sum_{z\in A} \mathrm{cedc}(z).$$

Proof. By assumption (ii) of Lemma 10.7, the non-regular points of D_j are exactly $z_1, ..., z_w$. Since these points form an isolated set, the divisor D_m has normal crossings by Lemma 10.3 and

$$\begin{aligned}\mathrm{cdc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F) = &\sum_{i=1}^{w} \mathrm{cdc}(z_i)\\ &+ \sum_{k\leq j: n\nmid \nu_k} [n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3)]\\ &+ \sum_{k\leq j: n\mid \nu_k} [-n + (n-1)\mu_k(\mu_k - 1)],\end{aligned}$$

where the second two sums run only over $k \leq j$, and μ_k, ν_k denote the multiplicities of the point y_k in the divisors D_{k-1} and $\mathrm{div}_0(f)$ on $\mathcal{Y}_{k-1}$ respectively.

For $k \leq j$, we have $\mu_k \geq 3$ by assumption (i), and since $n \geq 2$, it follows that $-n + (n-1)\mu_k(\mu_k - 1) \geq 0$ when $n \mid \nu_k$, and that $n - 2\gcd(\nu_k, n) + (n-1)\mu_k(\mu_k - 3) \geq 0$ when $n \nmid \nu_k$. It follows that

$$\mathrm{cdc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F) \geq \sum_{i=1}^{w} \mathrm{cdc}(z_i). \tag{10.9}$$

Similarly, we obtain

$$\mathrm{cedc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F) \geq \sum_{i=1}^{w} \mathrm{cedc}(z_i). \tag{10.10}$$

In the case $b = 0$ or $\gcd(b, n) = 1$, if we take $A = \{z_1, ..., z_w\}$, the result follows by substituting (10.9) and (10.10) respectively into the inequalities of Propositions 9.4 and 9.14.

Now consider the case $2 \leq \gcd(b, n) < n$. Let Γ denote the special fiber of $\mathcal{Y}_0$, let Γ'_j be its strict transform under $\mathcal{Y}_j \to \mathcal{Y}_0$, and let $e := \frac{n}{\gcd(b,n)}$ be its ramification index. Then $2 \leq e \leq \frac{n}{2}$.

Suppose $j = 0$, so that $\mathcal{Y}_j = \mathcal{Y}_0 = \mathbb{P}^1_{\mathcal{O}_K}$ satisfies assumption (ii) of Lemma 10.7. Since X is connected of genus $g \geq 1$, the divisor D_0 contains a horizontal component of ramification index $e' \neq e$, which intersects Γ at a point $y \in \{z_1, ..., z_w\}$. Since Γ is a regular irreducible component of D_0 of ramification index $e \neq n$, Proposition 10.6 implies

$$\mathrm{cedc}(y) \geq \mathrm{cdc}(y) \geq 2\frac{n}{e} - 2 = 2\gcd(b, n) - 2$$

If we take $A = \{z_1, ..., z_w\} \setminus \{y\}$, the result follows by substituting (10.9) and (10.10) into the inequalities of Proposition 9.4 and Corollary 9.14 respectively, and pairing off the terms $\mathrm{cdc}(y)$ and $\mathrm{cedc}(y)$ of (10.9) and (10.10) with the term $2 - 2\gcd(b, n)$ in Proposition 9.4 and Corollary 9.14.

Suppose instead $j > 0$. Then there is a point $y \in \mathcal{Y}_j$ at which Γ'_j intersects the exceptional divisor E of $\mathcal{Y}_j \to \mathcal{Y}_0$. Since the regular model $\mathcal{Y}_j$ of $\mathbb{P}^1_K$ has simple normal crossings by Lemma 7.5 and since Γ'_j and E are vertical divisors on $\mathcal{Y}_j$, then y is the intersection point of

Γ'_j with exactly one irreducible component of E, which is the strict transform $(E_k)'_j$ on $\mathcal{Y}_j$ of the exceptional divisor E_k of the blowup $\mathcal{Y}_k \to \mathcal{Y}_{k-1}$ for some $1 \le k \le j$.

If E_k is ramified, then y is a simple normal crossings point of D_j of multiplicity 2, hence $y \in \{z_1, ..., z_w\}$, and by Proposition 10.6,

$$\operatorname{cedc}(y) \ge \operatorname{cdc}(y) \ge 2\frac{n}{e} - 2 = 2\gcd(b, n) - 2$$

If we take $A = \{z_1, ..., z_w\} \setminus \{y\}$, the result follows as in the previous case.

Finally, if E_k is unramified, then $n \mid \nu_k$ and $\mu_k \ge 3$, so that the blowup $\mathcal{Y}_k \to \mathcal{Y}_{k-1}$ contributes a term of

$$-n + (n-1)\mu_k(\mu_k - 1) \ge 5n - 6 > 2\frac{n}{2} - 2 \ge 2\gcd(b, n) - 2$$

to $\operatorname{cdc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F)$. We can then improve inequalities 10.9 and 10.10 to

$$\operatorname{cdc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F) \ge 2\gcd(b, n) - 2 + \sum_{i=1}^{w} \operatorname{cdc}(z_i). \tag{10.11}$$

and

$$\operatorname{cedc}(\mathcal{Y}_m \to ... \to \mathcal{Y}_0, F) \ge 2\gcd(b, n) - 2 + \sum_{i=1}^{w} \operatorname{cedc}(z_i). \tag{10.12}$$

If we take $A = \{z_1, ..., z_w\}$, the result follows by substituting (10.11) and (10.12) respectively into the inequalities of Proposition 9.4 and Corollary 9.14. □

We now apply Proposition 10.8 and results of [Ste26] to deduce the conductor exponent-discriminant and conductor-discriminant inequalities for X (Corollary 1.9 = Corollary 10.14 and Theorem 1.4 = Corollary 10.19). Recall that, in light of (9.13), the conductor exponent-discriminant inequality is a consequence of the conductor-discriminant inequality. Since the conductor exponent-discriminant inequality is of independent interest and can be deduced without the full results of [Ste26], we first extract a simpler result from [Ste26] and prove the conductor exponent-discriminant inequality directly.

Proposition 10.13 ([Ste26, Theorem 1.4])**.** *Let y be as in Notation 10.1. Then* $\operatorname{cedc}(y) \ge 0$.

Corollary 10.14 (= Corollary 1.9)**.** *The conductor exponent-discriminant inequality holds for f. That is*

$$\varphi(X/K) \le (n-1)\nu_K(\operatorname{disc}_n(f)).$$

Proof. This follows immediately from Propositions 10.8 and 10.13. □

Unfortunately, the analog of Proposition 10.13 with cdc instead of cedc does not always hold, i.e., it is possible to have $\operatorname{cdc}(y) < 0$ for some y. But whenever this happens, it turns out that, after blowing up to obtain a normal crossings branch locus and then taking Hirzebruch–Jung resolutions of the tame cyclic quotient singularities, one can blow down sufficiently many -1-components to salvage the conductor-discriminant inequality. The proposition below gives the main result we need.

Let $\mathcal{Y}$ be a regular model of $\mathbb{P}^1_K$. Let $\mathcal{X}$ be the normalization of $\mathcal{Y}$ in $K(X)$, let D be the branch divisor of $\mathcal{X} \to \mathcal{Y}$, and let $y \in \mathcal{Y}$ be a multiplicity 2 point of D. Let $\mathcal{Y}_m \to ... \to \mathcal{Y}_0 = \mathcal{Y}$ be the local n-resolution of D at y, and let $\mathcal{X}_m$ be the normalization of $\mathcal{Y}_m$ in $K(X)$. Let $\mathcal{X}' \to \mathcal{X}$ and $\mathcal{X}'_m \to \mathcal{X}_m$ be the minimal resolutions of $\mathcal{X}$ and $\mathcal{X}_m$ respectively.

Proposition 10.15 ([Ste26, Theorem 1.6]). *Assume* $\operatorname{cdc}(y) < 0$. *Then the canonical morphism* $\pi : \mathcal{X}'_m \to \mathcal{X}'$ *contracts at least* $-\operatorname{cdc}(y)$ *irreducible components of the special fiber* $(\mathcal{X}'_m)_s$ *contained in the preimage of* y *on* $\mathcal{X}'_m$.

Remark 10.16. The condition $\operatorname{cdc}(y) < 0$ is only satisfied under very restrictive conditions on the degree of f, the orders of f along the components of D_0 containing y, and the length of the local resolution of D_0 at y. In fact, by [Ste26, Theorem 1.5], if $\operatorname{cdc}(y) < 0$, then for any point $x \in \mathcal{X}_0$ lying over y, the scheme $\operatorname{Spec} \hat{\mathcal{O}}_{\mathcal{X}_0,x}$ is a rational double point in the sense of [Lip69, Section 24], and the morphism $\mathcal{X}'_0 \to \mathcal{X}_0$ includes the standard ADE resolutions of all such points. The proof of Proposition 10.15 is technically involved and takes the bulk of [Ste26] — the proof of Proposition 10.13 is comparatively much simpler.

Example 10.17. Resume the notation of Example 10.4. Recall that

$$F_0 = F'_0 = F_y = D, \qquad (F_0 \bmod 3)^{\mathrm{red}} = F'_0,$$
$$F_1 = F'_1 + 2E_1, \qquad (F_1 \bmod 3)^{\mathrm{red}} = F'_1 + E_1,$$
$$F_2 = F'_2 + 2E'_1 + 3E_2, \qquad (F_2 \bmod 3)^{\mathrm{red}} = F'_2 + E'_1.$$

We compute $\operatorname{cdc}(y) := \operatorname{cdc}(\mathcal{Y}_2 \to \mathcal{Y}_1 \to \mathcal{Y}_0, D)$. In the notation of Definition 9.3, we have

$$\nu_1 = 2, \qquad \mu_1 = 2,$$
$$\nu_2 = 3, \qquad \mu_2 = 2,$$
$$\operatorname{ord}_f(F'_2) = 1, \qquad \operatorname{ord}_f(E'_1) = 2.$$

Let z_1 be the multiplicity 2 point of the branch divisor on $\mathcal{Y}_2$ that is the intersection of the two branch components F'_2 and E'_1. By Proposition 7.6 (iii), it follows that the associated ramification indices for the crossing at z_1 are $e_1 = e'_1 = 3$, and

$$\operatorname{HJL}(z_1) = L\left(3, -\left(\frac{1}{\gcd(1,3)}\right)^{-1} \cdot \frac{2}{\gcd(2,3)}\right) = L(3,1) = 1.$$

Plugging all these invariants into Definition 10.5, we get

$$\operatorname{cdc}(y) = 3 - 2\gcd(\nu_1, 3) + (3-1)\mu_1(\mu_1 - 3) - 3 + (3-1)\mu_2(\mu_2 - 1) + \left(\frac{3}{e_1} - 1\right) + \left(\frac{3}{e'_1} - 1\right) + \left(3 - \gcd\left(\frac{3}{e_1}, \frac{3}{e'_1}\right)(\operatorname{HJL}(z_1) + 1)\right) = -1.$$

It remains to show that the minimal resolution $\mathcal{X}'_2 \to \mathcal{X}_2$ of the normalization $\mathcal{X}_2$ of $\mathcal{Y}_2$ has at least one contractible component. The reader may refer to the proof of [Ste26, Proposition 6.1] in the case when $n = e_0 = 3$ and $j = 1$ for details; we simply summarize the results here. There are four irreducible components in $\mathcal{X}_2$ lying above the point y, of which three lie above the component E_2 in $\mathcal{Y}_2$, and one lies above the component E'_1 in $\mathcal{Y}_2$. Since $\operatorname{HJL}(z_1) = 1$, the minimal resolution of the unique singular point above z_1 adds another component, bringing the total number of components lying above y in $\mathcal{X}'_2$ to five.

On the other hand, the branch divisor D of the normalization $\mathcal{X} \to \mathcal{Y}$ is totally ramified, hence $\mathcal{X}$ contains a unique point x lying above y, for which $\hat{\mathcal{O}}_{\mathcal{X},x} \cong \mathcal{O}_K[[x,w]]/(w^3 - x^3 - \pi^2)$.

By Case III C. of [Lip69, Section 24] and the subsequent remark beginning with "Conversely...", we see Spec $\hat{\mathcal{O}}_{\mathcal{X},x}$ is a rational double point admitting a D_4 minimal resolution. In particular, if $\mathcal{X}' \to \mathcal{X}$ is the minimal resolution of $\mathcal{X}$, the preimage of y on $\mathcal{X}'$ contains four irreducible components of the special fiber $\mathcal{X}'_s$. It follows that the canonical morphism $\mathcal{X}'_2 \to \mathcal{X}'$ contracts $1 = -\operatorname{cdc}(y)$ irreducible component of the special fiber $(\mathcal{X}_2)'_s$ contained in the preimage of y.

Proposition 10.18. *Keep the notation of Proposition 10.8. Let $\mathcal{X}_{\min}$ be the minimal proper regular $\mathcal{O}_K$-model of X. Then*

$$\operatorname{Art}^+(\mathcal{X}'_m) - \operatorname{Art}^+(\mathcal{X}_{\min}) \geq -\sum_{z \in A : \operatorname{cdc}(z) < 0} \operatorname{cdc}(z).$$

Proof. Let $\pi : \mathcal{X}'_m \to \mathcal{X}_{\min}$ be the canonical contraction, and let N'_m and $N_{\min}$ denote the numbers of irreducible components of the (geometric) special fibers $(\mathcal{X}'_m)_s$ and $(\mathcal{X}_{\min})_s$ respectively. By Proposition 10.15, for each $z \in A$ with $\operatorname{cdc}(z) < 0$, the map π contracts at least $-\operatorname{cdc}(z)$ irreducible components of $(\mathcal{X}'_m)_s$ contained in the preimage of z_i on $\mathcal{X}'_m$. Since π is a composition of blow-downs of components of the special fiber and so creates no new components in the special fiber, we obtain

$$N'_m - N_{\min} \geq -\sum_{z \in A : \operatorname{cdc}(z) < 0} \operatorname{cdc}(z).$$

The result then follows from (9.13). ☐

Corollary 10.19 (= Theorem 1.4)**.** *The conductor-discriminant inequality holds for f, i.e.,*

$$\operatorname{Art}^+(\mathcal{X}_{\min}) \leq (n-1)\nu_K(\operatorname{disc}_n(f)).$$

Proof. This follows immediately from Propositions 10.8 and 10.18. ☐

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